INTEGRAL CALCULUS

Integral Calculus

FOR

HONOURS AND POST-GRADUATE STUDENTS

OF

All Indian Universities and for Various Competitive Examinations

BY

Prof. M. L. KHANNA

Associate Professor of Mathematics,

MEERUT COLLEGE, MEERUT

AUTHOR OF

A series of sixteen books for Post-Graduate classes; A series of eleven books for Degree classes; A series of seven books for Intermediate and Higher Secondary Examinations in Hindi and English; and A series of four books for Roorkee & Kharagpur Entrance Examinations

PUBLISHED BY

JAI PRAKASH NATH & Co.

EDUCATIONAL & LAW PUBLISHERS

MEERUT CITY

1967

BOOKS BY THE SAME AUTHOR

For Post-Graduate Classes—

1 Co-ordinate Solid Geometry
2 Theory of Equations
3. Vector Algebra
4 Vector Calculus
5 Determinants
6. Matrices
7 Spherical Trigonometry
8 Astronomy
9 Differential Calculus
10 Integral Calculus
11 Differential Equations
12 Partial Differential Equations
13 Dynamics of a Particle.
14 Statics
15 Attraction and Potential
16 Spherical Harmonics
17 Dynamics of Rigid bodies, (under preparation)

For Degree Classes—

18 Algebra
19. Trigonometry
20 Co-ordinate Geometry
21 Differential Calculus
22. Integral Calculus
23 Differential Equations.
24 Statics
25 Dynamics
26 Hydro-Statics
27 A Text Book of Integral Calculus
28. A Text Book of Differential Calculus

For Roorkee and Kharagpur Entrance Examinations—

29. Dynomics
30 Statics
31. Co-ordinate Geometry
32 Differential Calculus

Price Rs. 6.50 only.
1st Edition (Single) September, 1963
2nd Edition (Double) September, 1964
3rd Edition (Double) September, 1967

Published by
K. N. Gupta
for
JAI PRAKASH NATH & CO.
Meerut.

Printed by
I. D. Gupta
at the
PRAKASH PRINTING PRESS
Meerut.

PREFACE TO THE FIRST EDITION

It gives me great pleasure in bringing out my book on Integral Calculus for Post-Graduate and Honours students of various Indian universities

The style, presentation of theory and solution of various typical examples are in the same familiar form which the students have seen in my other thirty five books. I am confident that the students will like the present work which will prove useful to them.

Besides giving many solved examples, unsolved examples are also given for the practice of the students. At the end of the book, Sagar, Vikram and Agra University papers are attached, so that the students may know about the type of questions which are set in the paper.

Any suggestions for improvement of the book from any quarter will be highly appreciated.

M. L. Khanna

PREFACE TO THE THIRD EDITION

It gives me great pleasure in bringing out the third revised edition of the book The subject matter has been revised, and a few new questions have been added here and there. The references of various University papers have been given and in the end latest papers have also been added.

Warden's Bungalow
O'Donell Hostel
Meerut College

M. L. Khanna
Phone 2217

Where is What?

Chapter — Pages

LISTS OF FORMULAE

Chapter I

Definite Integrals

1. $\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

2. $\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}$

3. $\int_0^\infty \frac{\sin bx}{x} \, dx = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as b is $+$ive or $-$ive.

4. $\int_0^\infty e^{-x^2} \, dx = \frac{1}{2}\sqrt{\pi}$

5. $\int_0^\infty e^{-ax^2} \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{a}\right)}, \; a > 0$

6. $\int_0^\infty \frac{\cos mx}{a^2+x^2} \, dx = \frac{\pi}{2a} e^{-ma}$

7. $\int_0^\infty \frac{\sin mx}{x(a^2+x^2)} \, dx = \frac{\pi}{2a^2}(1-e^{-ma}).$

8. $\int_0^\infty \frac{x \sin mx}{a^2+x^2} = \frac{\pi}{2} e^{-ma}.$

9. $\int_0^\infty \frac{x^{2m}}{1+x^{2n}} \, dx = \frac{\pi}{2n} \operatorname{cosec} \left(\frac{2m+1}{2n} \pi\right).$

10. $\int_0^\infty \frac{x^{2m}}{1-x^{2n}} \, dx = \frac{\pi}{2n} \cot \left(\frac{2m+1}{2n} \pi\right).$

Deductions from 9 and 10.

11. $\int_0^\infty \frac{z^{a-1}}{1+z} \, dz = \pi \operatorname{cosec} a\pi, \int_0^\infty \frac{z^{a-1}}{1-z} \, dz = \pi \cot a\pi.$

12. $\int_0^\infty \frac{du}{1+u^{1/a}} \, du = a\pi \operatorname{cosec} a\pi, \; \int_0^\infty \frac{du}{1-u^{1/a}} = a\pi \cot a\pi.$

13. $\int_0^\infty \frac{du}{1+u^r}=\frac{\pi}{r}\operatorname{cosec}\frac{\pi}{r}$, $\int_0^\infty \frac{du}{1-u^r}=\frac{\pi}{r}\cot\frac{\pi}{r}$

14. $\int_0^\infty \frac{x^n+x^{-n}}{x+x^{-1}}\frac{dx}{x}=\frac{\pi}{2}\sec\frac{n\pi}{2}=\int_0^\infty \frac{x^n}{1+x^2}\,dx.$

15. $\int_0^\infty \frac{x^n-x^{-n}}{x-x^{-1}}\frac{dx}{x}=\frac{\pi}{2}\tan\frac{n\pi}{2}=\int_0^\infty \frac{x^n}{x^2-1}\,dx.$

16. $\int_0^\infty \frac{e^{az}+e^{-az}}{e^{\pi z}+e^{-\pi z}}\,dz=\frac{1}{2}\sec\frac{a}{2}.$

17. $\int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}}\,dz=\frac{1}{2}\tan\frac{a}{2}.$

Chapter II. **Beta and Gamma Functions**

1. $B(l, m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$

$$=\int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}\,dy=\int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}\,dy.$$

2. $\Gamma n=\int_0^1 e^{-x}x^{n-1}\,dx.$

3. $B(l, m)=B(m, l)=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}.$

4. $\Gamma n=(n-1)\Gamma(n-1)=(n-1)\,!$, $\Gamma 1=1$, $\Gamma 0=\infty$, $\Gamma(-n)=\infty$ and $\Gamma\frac{1}{2}=\sqrt{\pi}.$

5. $\int_0^\infty y^{n-1}e^{-ky}\,dy=\frac{\Gamma n}{k^n}.$

6. $\int_0^\infty e^{-y^{1/n}}\,dy=n\Gamma n=\Gamma(n+1).$

7. $\int_0^1 \log\left(\frac{1}{y}\right)^{n-1}dy=\Gamma n.$

8. $\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$ and $\Gamma(1+n)\Gamma(1-n)=\frac{n\pi}{\sin n\pi}.$

9. $$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

or $$\frac{[(p-1)(p-3)\ldots 2 \text{ or } 1][(q-1)(q-3)\ldots 2 \text{ or } 1]}{(p+q)(p+q-2)\ldots 2 \text{ or } 1}$$

$$\left[\frac{\pi}{2} \text{ only when both } p \text{ and } q \text{ are given}\right].$$

$$\int_0^{\pi/2} \sin^p \theta \, d\theta = \int_0^{\pi/2} \cos^p \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) . \Gamma\frac{1}{2}}{2\Gamma\left(\frac{p+2}{2}\right)}$$

$$= \frac{(p-1)(p-3)\ldots 2 \text{ or } 1}{p(p-2)\ldots 2 \text{ or } 1}.$$

10. $$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right) \quad \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\Gamma n}.$$

Chapter III **Dirichlet's Theorem**

I. $$\int x_1^{l_1-1} \, x_2^{l_2-1} \, dx_1 \, dx_2 = \frac{\Gamma(l_1) \, \Gamma(l_2)}{\Gamma(1+l_1+l_2)} \quad \text{where } x_1+x_2 < 1$$

$$= \frac{\Gamma(l_1) \, \Gamma(l_2)}{\Gamma(1+l_1+l_2)} . h^{l_1+l_2}$$

where $x_1+x_2 < h$.

II. $$\iint \ldots \int x_1^{l_1-1} . x_2^{l_2-1} \ldots x_n^{l_n-1} \, dx_1 \, dx_2 \ldots dx_n$$

$$= \frac{\Gamma l_1 \, \Gamma l_2 . \; . \Gamma l_n}{\Gamma(1+l_1+l_2+\ldots+l_n)}.$$

III. $$\iiint F(x+y+z) \, x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz$$

$$= \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(h) \, h^{l+m+n-1} \, dh,$$

where $h_1 < (x+y+z) < h_2$.

Chapter IV. **Volumes and Surfaces**

1. $V=\iiint dx\,dy\,dz$ in cartesian coordinates.

2. $\text{Mass}=\iiint \rho\,dx\,dy\,dz,$

$$V=\iiint r\,d\theta\,dr\,d\phi \text{ in polar co-ordinates.}$$

where $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, so that $x^2+y^2=r^2\sin^2\theta$ and $x^2+y^2+z^2=r^2$

3. $S=\iint \sec\gamma.dx\,dy$ in cartesian coordinates.

where $\sec\gamma=\sqrt{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]}$.

4. $S=\iint\left[r^2\left(\frac{\partial r}{\partial\phi}\right)^2+r^2\sin^2\theta\left(\frac{\partial r}{\partial\theta}\right)^2+r^4\sin^2\theta\right]^{1/2}d\theta\,d\phi.$

Chapter VI. **Fourier's Series**

1. Integral of $\cos mx$, $\sin mx$, $\cos mx\cos nx$, $\sin mx\sin nx$ is zero within limits $-\pi$ to π or 0 to 2π [m and n are integers].

2. Integral of $\cos^2 nx$ and $\sin^2 nx$ is π within limits $-\pi$ to π or $\pi/2$ if the limits be 0 to π, or 2π if the limits be 0 to 2π.

3. Fourier's Series. Interval $-\pi$ to π.

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where $a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(v)\,dv,$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(v)\cos nv\,dv,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(v)\sin nv\,dv$$

$$\therefore \quad f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)\, dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(v)\cos n\,(v-x)\, dv.$$

4. In the interval $(-\pi, \pi)$ if $f(x)$ is odd function, then $a_0=0$, $a_n=0$ and $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$=\frac{2}{\pi}\int_0^{\pi} f(v)\sin nv\, dv.$$

$$\therefore \quad f(x)=\sum_1^{\infty} b_n \sin nx$$

5. In the interval $(-\pi, \pi)$ if $f(x)$ be even function, then $b_n=0$ and $a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\, dx$ and $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

$$=\frac{2}{\pi}\int_0^{\pi} f(v)\cos nv\, dv.$$

$$\therefore \quad f(x)=a_0+\sum_1^{\infty}(a_n \cos nx)$$

6. **Fourier's Series, Interval $(0, \pi)$.**

Sine Series $\quad f(x)=\sum_1^{\infty} b_n \sin nx$

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$ as in (4).

Cosine Series $f(x)=a_0+\sum_1^{\infty} a_n \cos nx$

where $a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\, dx$ and $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$, as in (5)

7. **Fourier's Series, interval $(0, 2\pi)$.**

$$f(x)=a_0+\sum_1^{\infty}(a_n \cos nx+b_n \sin nx),$$

where $a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx.$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

In case $f(2\pi - x) = f(x)$, then

$$f(x) = a_0 + \sum_1^\infty a_n \cos nx.$$

In case $f(2\pi - x) = -f(x)$, then

$$f(x) = \sum_1^\infty b_n \sin nx.$$

where a_0, a_n, b_n have same values as written above.

In general, if $\alpha < x < 2\pi + \alpha$, then

$$f(x) = a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$

where $$a_0 = \frac{1}{2\pi}\int_\alpha^{2\pi+\alpha} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_\alpha^{2\pi+\alpha} f(x)\cos nx\,dx$$

and $$b_n = \frac{1}{\pi}\int_\alpha^{2\pi+\alpha} f(x)\sin nx\,dx.$$

8. **Fourier's Series. Interval $(-1, 1)$.**

$$f(x) = a_0 + \sum_1^\infty \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right),$$

where $$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx,$$

$$a_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{l}\,dx,$$

$$b_n = \frac{1}{l}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi x}{l}\,dx.$$

9. **Fourier's Series. Interval $(0, 1)$.**

Sine Series $f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}\,dx.$

where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

Cosine series $f(x)=a_0+\sum_1^\infty a_n\cos\frac{n\pi x}{l}\,dx,$

where $a_0=\frac{1}{l}\int_0^l f(x)\,dx,\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$

Chapter VII. **Convergence of Improper integrals.**

A. $\int_a^b f(x)\,dx$ **Range infinite and Integrand bounded. 1st kind.**

$$\int_a^\infty f(x)\,dx=\underset{X\to\infty}{\text{Lt}}\int_a^x f(x)\,dx=\text{finite} \quad \text{convergent}$$
$$=\pm\infty \quad \text{divergent}$$

$$\int_{-\infty}^b f(x)\,dx=\underset{X\to-\infty}{\text{Lt}}\int_x^b f(x)\,dx=\text{finite convergent}$$
$$=\pm\infty\ \text{divergent.}$$

$$\int_{-\infty}^\infty f(x)\,dx=\int_{-\infty}^a f(x)\,dx+\int_a^\infty f(x)\,dx.$$

Test.

I. Comparison Test. $\int_a^\infty f(x)\,dx$ is convergent if $\int_a^\infty \phi(x)\,dx$ is convergent and $f(x)<\phi(x)$ and $\int_a^\infty f(x)\,dx$ is divergent if $\int_a^\infty \phi(x)\,dx$ is divergent and $f(x)>\phi(x)$.

Also $\int_a^\infty \frac{dx}{x^n}$ **is convergent when n > 1 and divergent when n ≤ 1.**

II. μ**-Test. If** $\underset{X\to\infty}{\text{Lt}}\,x^\mu f(x)$ exists finitely, then $\int_a^\infty f(x)\,dx$ is convergent when $\mu>1$ and divergent when $\mu\le 1$

μ is chosen as "highest power of x in D^r − highest power of x in N^r".

III. Abel's Test. If $\int_a^\infty f(x)\,dx$ is convergent and $\phi(x)$ is monotonic and bounded for $x > a$, then $\int_a^\infty f(x)\,\phi(x)\,dx$ is also convergent

IV. *Dirichlet's Test.* *If $f(x)$ be bounded and monotonic and if $\underset{x\to\infty}{\text{Lt}} \int f(x) = 0$, then $\int_a^\infty f(x)\,\phi(x)\,dx$ converges,* provided $\left|\int_a^x \phi(x)\,dx\right|$ is bounded as x takes all finite values.

Absolute Convergence $\int_a^\infty f(x)\,dx$ *is said to converge absolutely if* $\int_a^\infty |f(x)|\,dx$ *is convergent.*

B. $\int_a^b f(x)\,dx$. **Range finite and integral unbounded.**

2nd kind.

$\int_a^b f(x)\,dx = \underset{\epsilon\to 0}{\text{Lt}} \int_{a+\epsilon}^b f(x)\,dx$ when $f(x)$ is unbounded at $x=a$.

$\int_a^b f(x)\,dx = \underset{\epsilon\to 0}{\text{Lt}} \int_a^{b-\epsilon} f(x)\,dx$, when $f(x)$ is unbounded at $x=b$.

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ when $f(x)$ is unbounded both at $x=a$ and $x=b$.

Tests.

Comparision Test. If $\underset{x\to a}{\text{Lt}} \dfrac{f(x)}{\phi(x)}$ be finite, then $\int_a^b f(x)\,dx$

and $\int_a^b \phi(x)\, dx$ both either converge or diverge together.

Auxiliary integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent when $n < 1$ and divergent when $n \geqslant 1$.

II. μ**-Test.** $\int_a^b f(x)\, dx$ when $x=a$ is a point of infinite discontinuity will be **convergent** if $\underset{x \to a+0}{\text{Lt}} (x-a)^\mu f(x)$ is finite and $\mu < 1$

If $\mu \geqslant 1$, then it is divergent. If however, above Lt is $\pm\infty$, then also divergent $\int_a^b f(x)\, dx$, when $x=b$ is a point of infinite discontinuity, then we shall evaluate $\underset{x \to b-0}{\text{Lt}} (b-x)^\mu f(x)$ and other conditions will be same as above.

CHAPTER I

D EFINITE INTEGRALS

§ 1. **Def.** We know that if $\int f(x)\,dx = F(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where a and b are called the lower and upper limits and $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ between the above limits. It represents the area under the curve $y=f(x)$, x-axis and the ordinates drawn at $x=a$, $x=b$. (See author's Integral Calculus Chapter VIII).

Properties of Defiaitc Integrals.

Below we shall give the important properties of definite integrals. For their proofs students should see author's book on Integral Calculus for B.Sc. Chapter VI P. 238.

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$

In other words, the above theorem states that in the case of definite integrals, if the limits remains the same and the function remains the same but the variable changes, then there is no change in the value of the definite integral,

i.e. $$\int_a^b x^2\,dx = \int_a^b t^2\,dt = \frac{1}{3}(b^3 - a^3).$$

2. $\int_a^b f(x)\,dx = -\int_b^a$ $\quad$ $(x.$

In other words it means that if we interchange the of a definite integral, then we change the sign as well.

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$ (Karnatak 64)

In general,

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots \int_{c_n}^b f(x)\,dx.$$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$ (Imp.)

L.H.S. $= \int_0^a f(x)\,dx.$ Put $x = a - t$; $\therefore$ $dx = -dt,$

Also when $x = a$, $t = 0$ and when $x = 0$, $t = a$.

$$\therefore \quad \text{L.H.S.} = \int_a^0 f(a-t)(-dt) = \int_a^0 f(a-t)\,dt. \quad \text{(Prop. 2)}$$

$$= \int_0^a f(a-x)\,dx. \quad \text{(Prop. 1)} \qquad \text{Proved.}$$

In other words, it means that in the case of definite integral in which the lower and upper limits are 0 to a respectively, we can alway write $a-x$ for x and the value of definite integral is not affected. The limits of integration however remain unchanged,

e.g. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n \left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x\,dx.$

5. $\int_{-a}^a f(x)\,dx = 0$ if $f(x)$ is an odd function of x,

i.e. $f(-x) = -f(x).$

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function of x, i.e. $f(-x) = f(x).$

$$\text{L.H.S} = \int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx.$$

(Prop. 3)

In the first integral in R.H.S. put $x = -t$ so that $dx = -dt$ and adjust the limits.

$$\therefore \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)\,(-dt) = \int_{0}^{a} f(-t)\,dt \quad \textbf{(Prop. 2)}$$

$$= \int_{0}^{a} f(-x)\,dx. \quad \textbf{(Prop. 1)}$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 0 \text{ if } f(x) \text{ is odd, } i.e.\ f(-x) = -f(x)$$

$$= 2\int_{0}^{a} f(x)\,dx \text{ if } f(x) \text{ is even,}$$

$$i.e.\ f(-x) = f(x).$$

6. $\int_{0}^{2a} \mathbf{f(x)\,dx} = 2\int_{0}^{a} \mathbf{f(x)\,dx}$ **if** $\mathbf{f(2a-x) = f(x)}$

$\mathbf{= 0}$ **if** $\mathbf{f(2a-x) = -f(x)}$.

$$\text{L.H S.} = \int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx. \quad \textbf{(Prop. 3)}$$

In 2nd integral in R.H.S. put $x = 2a - t$; $\therefore\ dx = -dt$ and adjust the limits

$$\therefore \int_{0}^{2a} f(x)\,dx = \int_{a}^{0} f(2a-t)\,(-dt) = \int_{0}^{a} f(2a-t)\,dt \quad \textbf{(Prop. 2)}$$

$$= \int_{0}^{a} f(2a-x)\,dx. \quad \textbf{(Prop. 1)}$$

$$\therefore \int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x) + \int_{0}^{a} f(2a-x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)$$

$$= 0 \text{ if } f(2a-x) = -f(x). \quad \textbf{Proved.}$$

Applications.

$$\int_{0}^{\pi} \sin^{n} x\,dx = 2\int_{0}^{\pi/2} \sin^{n} x\,dx.$$

$\therefore\ \sin^{n}(\pi - x) = \sin^{n} x,\quad i.e.\ f(2a-x) = f(x).$

$$\int_0^{\pi} \cos^n x\, dx = 2\int_0^{\pi/2} \cos^n x\, dx \text{ if } n \text{ is even}$$
$$= 0 \text{ if } n \text{ is odd.}$$

$\therefore\ \cos^n(\pi - x) = (-\cos x)^n = \cos^n x$ or $-\cos^n x$ according as n is even or odd,
i e. $f(2a-x)=f(x)$ or $-f(x)$ according as n is even or odd.

Similarly, $\int_0^{\pi} \sin^m x \cos^n x\, dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ or 0 according as n the power of cos x is even or odd, whatever the power of sin x may be.

Below we shall give certain questions explaining the application of above properties.

Exercise 1

Ex. 1. $\int_0^{\pi/2} \log \sin x\, dx = \frac{\pi}{2} \log \frac{1}{2} = \int_0^{\pi/2} \log \cos x\, dx.$

Let $I = \int_0^{\pi/2} \log \sin x\, dx.$

$$I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \log \cos x\, dx.$$

(Prop. 4)

$$\therefore\ 2I = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \cos x\, dx$$

$$I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$I = \int_0^{\pi/2} \log \sin x \cos x\, dx = \int_0^{\pi/2} \log \frac{\sin 2x}{2}\, dx$$

$$\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = \frac{\pi}{2} \log \tfrac{1}{2} = -\frac{\pi}{2} \log 2.$$

$$=\int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx.$$

In the 1st integral put $2x=t$ and adjust the limits.

$$\therefore \quad 2I=\frac{1}{2}\int_0^{\pi} \log \sin t\, dt-\Big[x \log 2\Big]_0^{\pi/2}$$

$$=\frac{1}{2}.2\int_0^{\pi/2} \log \sin t\, dt \text{ (Prop. 6)}-\frac{\pi}{2}\log 2$$

$$=\int_0^{\pi/2} \log \sin x\, dx \text{ (Prop. 1)}-\frac{\pi}{2}\log 2=I-\frac{\pi}{2}\log 2.$$

$\therefore$ Taking I in the L.H.S., we get

$$I=-\frac{\pi}{2}\log 2=\frac{\pi}{2}\log\frac{1}{2}.$$

Hence remember that

$$\int_0^{\pi/2} \log \sin x\, dx=\int_0^{\pi/2} \log \cos x\, dx=\frac{\pi}{2}\log\frac{1}{2}.$$

Again $\log \operatorname{cosec} x=\log (\sin x)^{-1}=-\log \sin x$

and $\quad \log \sec x=\log (\cos x)^{-1}=-\log \cos x.$

$$\therefore \quad \int_0^{\pi/2} \log \sec x\, dx=\int_0^{\pi/2} \log \operatorname{cosec} x\, dx$$

$$=-\frac{\pi}{2}\log\frac{1}{2}=\frac{\pi}{2}\log 2.$$

Again

$$\int_0^{\pi/2} \log \tan x\, dx=\int_0^{\pi/2} \log \sin x\, dx-\int_0^{\pi/2} \log \cos x\, dx=0$$

and $$\int_0^{\pi/2} \log \cot x\, dx=\int_0^{\pi/2} \log \cos x\, dx-\int_0^{\pi/2} \log \sin x\, dx=0$$

$$\int_0^1 \log \sin\left(\frac{\pi}{2}y\right) dx=\log\frac{1}{2}. \quad \text{Put } \frac{\pi}{2}y=z \text{ etc.}$$

$$\therefore \quad I=\int_0^{\pi/2} (\log \sin z)\left(\frac{2}{\pi}dz\right)=\frac{2}{\pi}\frac{\pi}{2}\log\frac{1}{2}=\log\frac{1}{2}.$$

Note. *The above results may be taken for granted in questions though in the examination you should deduce them.*

Ex. 2. *Evaluate the following definite integrals* :—

(a) $\int_0^1 \frac{\sin^{-1} x}{x} dx$. (b) $\int_0^{\pi/2} \theta \cot \theta \, d\theta$.

(c) $\int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta \, d\theta$. (d) $\int_0^{\infty} (\cot^{-1} x)^2 \, dx$.

$I=\int_0^1 \frac{\sin^{-1} x}{x} \, dx = \sin \theta$ and adjust the limits.

$I=\int_0^{\pi/2} \frac{\theta}{\sin \theta} (\cos \theta) \, d\theta = \int_0^{\pi/2} \theta \cot \theta \, d\theta$. [It is part (b)].

Now integrate by parts.

$$I=\Big[\theta \log \sin \theta\Big]_0^{\pi/2} - \int_0^{\pi/2} \log \sin \theta \, d\theta$$

$$=0-\frac{\pi}{2} \log \frac{1}{2} \text{ (Q. 1)} = \frac{\pi}{2} \log 2.$$

Note. V. Imp. *$\theta \log \sin \theta$, when $\theta=0$ takes indeterminate form $0 \times \infty$.*

$$\underset{\theta \to 0}{\text{Lt}} \theta \log \sin \theta = \underset{\theta \to 0}{\text{Lt}} \frac{\log \sin \theta}{\frac{1}{\theta}} = \underset{\theta \to 0}{\text{Lt}} \frac{\frac{\cos \theta}{\sin \theta}}{\frac{1}{-\theta^2}}$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{\theta^2 \cos \theta}{-\sin \theta} = \underset{\theta \to 0}{\text{Lt}} \frac{2\theta \cos \theta - \theta^2 \sin \theta}{-\cos \theta} = \frac{0}{-1} = 0.$$

(c) $I=\int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta \, d\theta$.

Integrate by parts

$=\Big[\theta^2 (-\cot \theta)\Big]_0^{\pi/2} + 2 \int_0^{\pi/2} \theta \cot \theta \, d\theta$

$=0+2.\left[\frac{\pi}{2} \log 2\right]$ (Part a and b) $= \pi \log 2$.

Note. *You should show as above that* $\underset{\theta \to 0}{Lt} \theta^2 \cot \theta = 0$.

(d) $\int_0^{\infty} (cot^{-1} x)^2 \, dx.$

Put $x = \cot \theta$; $\therefore \quad dx = -\operatorname{cosec}^2 \theta \, d\theta.$

$I = \int_{\pi/2}^{0} \theta^2 (-\operatorname{cosec}^2 \theta) \, d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta \, d\theta$ etc. (part c)

$= \pi \log 2.$ Ans.

(e) $\int_0^{\pi} log\,(1+cos\, x)\, dx.$

$I = \int_0^{\pi} \log (1+\cos x) \, dx.$

$I = \int_0^{\pi} \log \{1+\cos (\pi - x)\} \, dx = \int_0^{\pi} \log (1-\cos x) \, dx$ by IV.

$2I = \int_0^{\pi} [\log (1+\cos x) + \log (1-\cos x)] \, dx$

$= \int_0^{\pi} \log (1-\cos^2 x) \, dx = \int_0^{\pi} \log \sin^2 x \, dx$

or $\quad 2I = 2 \int_0^{\pi} \log \sin x \, dx = 2.2 \int_0^{\pi/2} \log \sin x \, dx$ by VI.

$\therefore \quad I = 2 . \frac{\pi}{2} \log \frac{1}{2} = \pi \log \frac{1}{2}.$

(f) $\int_0^{\infty} \frac{log\,(1+x^2)}{1+x^2} \, dx = \pi \log 2.$

Put $x = \tan \theta$; $\therefore 1+x^2 = \sec^2 \theta$; $\therefore I = \int_0^{\pi/2} 2 \log \sec \theta \, d\theta.$

Ex. 3. *Show that the value of each of the following definite integrals is* $\pi/4$:—

(a) $\int_0^{\pi/2} \frac{sin\, x}{sin\, x + cos\, x} \, dx.$ (b) $\int_0^{\pi/2} \frac{cos\, x}{sin\, x + cos\, x} \, dx.$

*(c) $\int_0^{\pi/2} \frac{\sqrt{(sin\, x)}}{\sqrt{(sin\, x)} + \sqrt{(cos\, x)}} \, dx.$ (d) $\int_0^{\pi/2} \frac{dx}{1 + tan\, x}.$

All the *m* part of Q. 3 are very important from the point of view of examination. All are based on property 4.

(e) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$. (f) $\int_0^{\pi/2} \frac{dx}{1+\cot x}$.

(g) $\int_0^{\pi/2} \frac{dx}{1+\sqrt{(\cot x)}}$. (h) $\int_0^{\pi/2} \frac{\tan x\, dx}{1+\tan x}$

(i) $\int_0^{\pi/2} \frac{\cot x}{1+\cot x}\, dx$. (j) $\int_0^{\pi/2} \frac{\sqrt{(\tan x)}\, dx}{1+\sqrt{(\tan x)}}$.

(k) $\int_0^{\pi/2} \frac{\sqrt{(\cot x)}}{1+\sqrt{(\cot x)}}\, dx$. *(l) $\int_0^{a} \frac{dx}{x+\sqrt{(a^2-x^2)}}$

(m) $\int_0^{\infty} \frac{x\, dx}{(1+x)(1+x^2)}$.

(a) $I=\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\, dx.$ ...(1)

$$\therefore \quad I=\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) dx}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)} \quad \text{(Prop. 4)}$$

$$=\int_0^{\pi/2} \frac{\cos x\, dx}{\cos x+\sin x}, \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi/2} \frac{(\sin x+\cos x)}{(\sin x+\cos x)} dx$$

or

$$2I=\int_0^{\pi/2} dx=\Big[x\Big]_0^{\pi/2}=\frac{\pi}{2}; \quad \therefore \quad I=\frac{\pi}{4}.$$

In other parts change $\tan x$ and $\cot x$ in terms of $\sin x$ and $\cos x$ and proceed as above.

(k) $I=\int_0^{\pi/2} \frac{\sqrt{(\cot x)}}{1+\sqrt{(\cot x)}} dx$

$$=\int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\, dx.$$

$$I=\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}\, dx. \quad \text{(Property 4)}$$

$$2I=\int_0^{\pi/2} 1.dx=\frac{\pi}{2}\ ; \quad \therefore \quad I=\frac{\pi}{4}.$$

(l) $I=\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}}$.

Put $x=a \sin \theta$ and adjust the limits.

$$I=\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta+\cos\theta} \text{ etc.} =\frac{\pi}{4}.$$

(m) $\int_0^{\infty} \frac{x\, dx}{(1+x)(1+x^2)}$.

Put $x=\tan\theta$ and adjust the limits.

$$I=\int_0^{\pi/2} \frac{\tan\theta \sec^2\theta\, d\theta}{(1+\tan\theta)(\sec^2\theta)}=\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta+\sin\theta} \text{ etc.} =\frac{\pi}{4}.$$

Ex. 4. (a) $\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx=\frac{\pi}{8}\log 2.$ (Agra 64)

(b) $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta=\frac{\pi}{8}\log 2.$

(a) and (b). Put $x=\tan\theta$ and adjust the limits.

$$I=\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta=\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta.$$

$$=\int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta \qquad \text{(Property 4)}$$

$$=\int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$=\int_0^{\pi/4} \log \frac{2}{1+\tan\theta}\, d\theta$$

$$=\int_0^{\pi/4} \log 2\, d\theta-\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta.$$

$$I=\Big[\theta \log 2\Big]_0^{\pi/4}-I \quad \text{or} \quad 2I=\frac{\pi}{4}\log 2\ ; \quad \therefore \quad I=\frac{\pi}{8}\log 2.$$

Ex. 5. (a) $\int_0^1 \frac{\log x\, dx}{\sqrt{(1-x^2)}}=\frac{\pi}{2}\log\frac{1}{2}.$

*(b) $\int_0^\infty \log\left(x+\frac{1}{x}\right) \frac{dx}{1+x^2}=\pi \log 2.$

(a) Put $x=\sin\theta$ and see Q. 1.

(b) Put $x=\tan\theta$; $\therefore\ I=\int_0^{\pi/2} \log \frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\,d\theta$

or $$I=\int_0^{\pi/2} (\log \sec\theta+\log \operatorname{cosec}\theta)\,d\theta$$

$$=\frac{\pi}{2}\log 2+\frac{\pi}{2}\log 2=\pi\log 2.$$

Note. (Imp.) Removal of x.

Suppose you know the integral of $\phi(x)$: then in order to evaluate the integral of $x\,\phi(x)$, we try to remove the factor x. This is done by the help of Prop. no. 4, provided $\phi(x)$ does not change when x is replaced by $(a-x)$;

i.e. if $\phi(a-x)=\phi(x)$,

i.e. $I=\int_0^a x\,\phi(x)\,dx$

$$=\int_0^a (a-x)\,\phi(a-x)\,dx \text{ (Prop. 4)}=\int_0^a (a-x)\,\phi(x)\,dx.$$

$$\therefore\ 2I=\int_0^a (x+a-x)\,\phi(x)\,dx=a\int_0^a \phi(x)\,dx.$$

$$\therefore\ I=\frac{a}{2}\int_0^a \phi(x)\,dx, \text{ provided } \phi(a-x)=\phi(x). \quad \ldots(1)$$

It is to be noted that x is eliminated only when $\phi(x)$ remains unchanged when $(a-x)$ is Put in place of x. Hence before applying the above method it should be observed that $\phi(a-x)=\phi(x)$. In the examination students should deduce step I as above.

Ex. 6. (a) $\int_0^\pi x\sin^6 x\cos^2 x\,dx$. (b) $\int_0^\pi x\sin^8 x\,dx$.

(a) Here, if we replace x by $\pi-x$ in $\sin^6 x\cos^2 x$, it does not change.

$$\therefore \quad I=\int_0^{\pi} (\pi-x) \sin^6 x \cos^2 x \, dx \quad \text{(Prop. 4)}.$$

$$\therefore \quad 2I=\int_0^{\pi} (x+\pi-x) \sin^6 x \cos^2 x \, dx=\pi \int_0^{\pi} \sin^6 x \cos^2 x \, dx.$$

$$2I=\pi.2 \int_0^{\pi/2} \sin^6 x \cos^2 x \, dx. \quad \text{(Prop. 6)}.$$

$$\therefore \quad I=\pi.\frac{5.3.1.1}{8.6.4.2}.\frac{\pi}{2}=\frac{5\pi^2}{256}.$$

(b) $\int_0^{\pi} x \sin^5 x \, dx=\frac{8}{15}\pi.$

Remove x in (a).

$$2I=\pi \int_0^{\pi} \sin^5 x \, dx=2\pi \int_0^{\pi/2} \sin^5 x \, dx$$

or $$2I=2\pi.\frac{4.2}{5.3}; \quad \therefore \quad I=\frac{8\pi}{15}.$$

Note. $\int_0^{\pi} x \cos^3 x \, dx$ *cannot be evaluated by removing* x *as before because* $\cos^3 x$ *changes sign when* x *is replaced by* $\pi-x$.

Ex. 7. (a) $\int_0^{\pi} \frac{x \, dx}{a^2 \cos^2 x+b^2 \sin^2 x}.$

*(b) $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx.$

(c) $\int_0^{\pi} \frac{x \tan x \, dx}{\sec x+\tan x}.$

(d) $\int_0^{\pi} \frac{x \, dx}{a^2-\cos^2 x}, \; a > 1.$

(d_1) $\int_0^{\pi} \frac{x \, dx}{1+\cos^2 x}=\frac{\pi^2}{2\sqrt{2}}.$

*(e) $\int_0^{\pi} x \log \sin x \, dx.$

(f) $\int_0^{\pi} xf(\sin x)\, dx.$

(g) $\int_0^{\pi} \frac{x \tan x\, dx}{\sec x+\cos x}.$

(h) $\int_0^{\pi} \frac{x\, dx}{(a^2 \cos^2 x+b^2 \sin^2 x)^2}=\frac{\pi^2 (a^2+b^2)}{4a^3b^3}.$

(i) $\int_0^{\pi} \frac{x \sin x\, dx}{1+\sin x}.$

All the above questions are of the type $\int_0^a x\, \phi(x)$, where $\phi(x)$ does not change when x is replaced by $(a-x)$ and hence should be done as shown in Note given after Q. 5,

i.e. $I=\int_0^a x\, \phi(x)\, dx$, then $I=\frac{a}{2}\int_0^a \phi(x)\, dx$ provided $\phi(a-x)=\phi(x).$

(a) $I=\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x+b^2 \sin^2 x}.$ Here $\phi(\pi-x)=\phi(x).$

$\therefore \quad I=\int_0^{\pi} \frac{(\pi-x)\, dx}{a^2 \cos^2 x+b^2 \sin^2 x}.$

Adding, we get

$$2I=\int_0^{\pi} \frac{\pi\, dx}{a^2 \cos^2 x+b^2 \sin^2 x}$$

or

$$I=2.\frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x+b^2 \sin^2 x} \text{ (Prop. 6).}$$

$$\therefore \quad I=\pi \int_0^{\pi/2} \frac{\sec^2 x\, dx}{a^2+b^2 \tan^2 x}=\frac{\pi}{b}.\frac{1}{a} \tan^{-1} \left[\frac{b \tan x}{a}\right]_0^{\pi/2}$$

$$=\frac{\pi}{ab}\left[\frac{\pi}{2}-0\right]=\frac{\pi^2}{2ab}.$$ **Ans.**

All the parts of Q. 7 are important from examination point of view.

(b) *Proceeding as in part (a),*

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x\,dx}{1+\cos^2 x}=-\frac{\pi}{2}\int_0^{\pi}\frac{-\sin x\,dx}{1+\cos^2 x}$$

$$=-\frac{\pi}{2}\left[\tan^{-1}\cos x\right]_0^{\pi}$$

$$=-\frac{\pi}{2}\left[\tan^{-1}(-1)-\tan^{-1}1\right]=-\frac{\pi}{2}\left[-\frac{\pi}{4}-\frac{\pi}{4}\right]=\frac{\pi^2}{4}.\quad \textbf{Ans.}$$

(c) $$I=\int_0^{\pi}\frac{x\tan x\,dx}{\sec x+\tan x}.$$

$$=\int_0^{\pi}\frac{(\pi-x)(-\tan x)}{(-\sec x)+(-\tan x)}\,dx. \qquad \text{(Prop. 4)}$$

$$=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}\,dx.$$

Adding, we get

$$2I=\pi\int_0^{\pi}\frac{\tan x\,dx}{\sec x+\tan x}.$$

$$\therefore\quad I=\frac{\pi}{2}\int_0^{\pi}\frac{\tan x(\sec x-\tan x)}{\sec^2 x-\tan^2 x}\,dx$$

or $$I=\frac{\pi}{2}\int_0^{\pi}\sec x\tan x-(\sec^2 x-1)\,dx$$

$$=\frac{\pi}{2}\left[\sec x-\tan x+x\right]_0^{\pi}=\frac{\pi}{2}\,(\pi-2).$$

(d) $$I=\int_0^{\pi}\frac{x\,dx}{a^2-\cos^2 x}\ (a>1).$$ Proceeding as before.

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2-\cos^2 x}=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2 x\,dx}{a^2(1+\tan^2 x)-1}.$$

$$\frac{\pi}{2a}\int_0^{\pi}\frac{a\sec^2 x\,dx}{a^2\tan^2 x+\{\sqrt{(a^2-1)}\}^2}$$

$$=\frac{\pi}{2a}\cdot\frac{1}{\sqrt{(a^2-1)}}\left[\tan^{-1}\frac{a\tan x}{\sqrt{(a^2-1)}}\right]_0^{\pi}$$

For $\int\frac{dx}{a+b\cos^2 x+c\sin^2 x}$ divide above and below by $\cos^2 x$ and change in terms of $\tan x$ and then put $\tan x=t$.

$$=\frac{\pi}{2a\sqrt{(a^2-1)}}(\pi-0)=\frac{\pi^2}{2a\sqrt{(a^2-1)}}.\qquad \text{Ans.}$$

(e) $I=\int_0^\pi x\log\sin x\,dx.$ Proceeding as before,

$$=\frac{\pi}{2}\int_0^\pi \log\sin x\,dx=\frac{\pi}{2}.2\int_0^{\pi/2}\log\sin x\,dx.$$

(Prop. 6)

$$=\pi.\frac{\pi}{2}\log\frac{1}{2}=\frac{\pi^2}{2}\log\frac{1}{2}.\ (\text{Ex. 1, P. 4}).\qquad \text{Ans.}$$

(f) $I=\int_0^\pi xf(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx.$

(g) *The question is easily seen to be* $I=\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$

which we have done in part (b).

(h) $I=\int_0^\pi \frac{x\,dx}{(a^2\cos^2 x+b^2\sin^2 x)^2};\ \because\ \phi(a-x)=\phi(x).$

$\therefore\ I=\int_0^\pi \frac{dx}{(a^2\cos^2 x+b^2\sin^2 x)^2}.$ (See P. 10)

Dividing above and below by $\cos^4 x$, we get

$$I=\frac{\pi}{2}\int_0^\pi \frac{\sec^2 x\sec^2 x\,dx}{(a^2+b^2\tan^2 x)^2}.$$

Now put $b\tan x=a\tan\theta$; $\therefore\ \sec^2 x\,dx=\frac{a}{b}\sec^2\theta\,d\theta.$

$$\therefore\quad I=\frac{\pi}{2}\int_0^\pi \frac{\left(1+\frac{a^2}{b^2}\tan^2\theta\right).\frac{a}{b}\sec^2\theta\,d\theta}{a^4\sec^4\theta}$$

$$=\frac{\pi}{2a^3b^3}\int_0^\pi (b^2\sin^2\theta+a^2\cos^2\theta)\,d\theta$$

$$=\frac{\pi}{2a^3b^3}.2\int_0^{\pi/2}(b^2\sin^2\theta+a^2\cos^2\theta)\ (\text{Prop. VI})$$

$$=\frac{\pi}{a^3b^3}\left[b^2.\frac{1}{2}.\frac{\pi}{2}+a^2.\frac{1}{2}.\frac{\pi}{2}\right]=\frac{\pi^2}{4a^3b^3}(a^2+b^2).$$

(i) Proceeding as usual, $I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\sin x}\,dx$

or $I=\frac{\pi}{2}\left(1-\frac{1}{1+\sin x}\right)dx=\frac{\pi}{2}\left[\pi-\int\frac{dx}{(\cos x/2+\sin x/2)^2}\right]$

$=\frac{\pi}{2}\left[x-2\int_0^{\pi}\frac{\frac{1}{2}.\sec^2 x/2\,dx}{(1+\tan x/2)^2}\right]=\frac{\pi}{2}\left[\pi-2\left(-\frac{1}{1+\tan x/2}\right)_0^{\pi}\right]$

$=\frac{\pi}{2}\left[\pi+2\left(\frac{1}{\infty}-1\right)\right]=\frac{\pi}{2}(\pi-2).$

Ex. 8. (a) $\int_0^{\pi/2}\frac{x\,dx}{\sin x+\cos x}=\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1).$

*(b) $\int_0^{\pi/2}\frac{\sin^2 x\,dx}{\sin x+\cos x}=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$

(a) $I=\int_0^{\pi/2}\frac{\left(\frac{\pi}{2}-x\right)dx}{\cos x+\sin x}.$ (Prop. 4)

$\therefore\ 2I=\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$

or $2I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}dx}{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}=\frac{\pi}{2}I'$ say.

$\therefore\ I=\frac{\pi}{4}I',$ ...(1)

where $I'=\int_0^1\frac{2\,dt}{(\sqrt{2})^2-(t-1)^2}$, where $t=\tan\frac{x}{2}$ etc.

$=2.\frac{1}{2\sqrt{2}}\left[\log\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right]_0^1=0-\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}-1}{\sqrt{2}+1}$

For $\int\frac{dx}{a+b\sin x+c\cos x}$ change $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ and then put $\tan\frac{x}{2}=t$ etc.

$$=\frac{1}{\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{1}{\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{2-1}=\frac{2}{\sqrt{2}}\log(\sqrt{2}+1).$$

$$\therefore\quad I=\frac{\pi}{2}I'=\frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1).$$

(b) Use Prop. 4 and proceed as above.

Ex. 9. (a) $\int_{-1}^{1}\frac{x^2\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx=0.$ (b) $\int_{-1}^{1}\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx=2.$

(c) $\int_{-a}^{a}x\sqrt{(a^2-x^2)}\,dx=0.$

Put $x=a\sin\theta$ and make use of Prop. no. 5.

Ex. 10. (a) $\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\sin x\cos x}\,dx=0.$

(b) $\int_0^{\pi/2}\sin 2x\log\tan x\,dx=0.$

(c) $\int_0^{\pi/2}(\sin x-\cos x)\log(\sin x+\cos x)\,dx=0.$

(d) $\int_0^{1}\log\left(\frac{1}{x}-1\right)dx=0.$

By using Prop. 4, we shall get $I=-I$, i.e. $2I=0$; $\therefore I=0$.

Ex. 11. *Prove that*

$$\int_0^{\pi/2}f(\sin 2x)\sin x\,dx=\int_0^{\pi/2}f(\sin 2x)\cos x\,dx$$

$$=\sqrt{2}\int_0^{\pi/4}f(\cos 2x)\cos x\,dx,$$

$$I=\int_0^{\pi/2}f(\sin 2x)\sin x\,dx.$$

$\therefore$ By prop. (4), $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, we get

$$I=\int_0^{\pi/2}f(\sin 2x)\cos x\,dx.\quad\text{Adding the two,}$$

$$2I=\int_0^{\pi/2}f(\sin 2x)(\sin x+\cos x)\,dx.$$

$$=\sqrt{2}\int_0^{\pi/2} f(\sin 2x)\sin\left(x+\frac{\pi}{4}\right)dx.$$

Now put $x+\frac{\pi}{4}=\frac{\pi}{2}-\theta$, *i.e.* $x=\frac{\pi}{4}-\theta$, and adjust the limits.

Also $2x=\frac{\pi}{2}-2\theta$; $\therefore$ $\sin 2x=\cos 2\theta$.

$$2I=-\sqrt{2}\int_{\pi/4}^{-\pi/4} f(\cos 2\theta)\cos\theta\, d\theta$$

$$=\sqrt{2}\int_{-\pi/4}^{\pi/4} f(\cos 2\theta)\cos\theta\, d\theta. \qquad \textbf{(Prop. 2)}$$

or $$2I=2\sqrt{2}\int_0^{\pi/4} f(\cos 2\theta)\cos\theta\, d\theta \qquad \textbf{(Prop. 5)}$$

even function.

Ex. 12. (a) *Show that*

$$\int_0^{\pi}\frac{x^2\sin 2x\sin(\frac{1}{2}\pi\cos x)}{2x-\pi}\,dx=\frac{8}{\pi}.$$

Put $x=\pi/2-t$, $\therefore$ $dx=-dt$, and adjust the limits.

$$\therefore\quad I=\int_{\pi/2}^{-\pi/2}\frac{(\pi/2-t)^2\sin(\pi-2t)\sin\{\pi/2\cos(\pi/2-t)\}(-dt)}{\pi-2t-\pi}$$

$$=-\int_{-\pi/2}^{\pi/2}\frac{(\frac{1}{4}\pi^2-\pi t+t^2)\sin 2t\sin(\frac{1}{2}\pi\sin t)\,dt}{2t}$$

(Prop. 2)

Now $\dfrac{(\pi^2/4+t^2)\sin 2t\sin(\frac{1}{2}\pi\sin t)}{2t}$ is an odd function of of t because if we replace t by $-t$, then the function changes sign, $\because$ $\sin(-\theta)=-\sin\theta$, and hence by prop. 5, we have

$$I=-2\int_0^{\pi/2}\frac{-\pi t\sin 2t\sin(\frac{1}{2}\pi\sin t)\,dt}{2t}$$

$$=\pi\int_0^{\pi/2}2\sin t\cos t\sin\left(\frac{\pi}{2}\sin t\right)dt.$$

Put $\frac{1}{2}\pi\sin t=z$; $\therefore$ $\frac{1}{2}\pi\cos t\,dt=dz$.

$$\therefore \quad I=\pi \int_0^{\pi/2} 2 . \frac{2z}{\pi} \sin z \left(\frac{2}{\pi} dz\right)=\frac{8}{\pi} \int_0^{\pi/2} z \sin z \, dz$$

$$=\frac{8}{\pi}\left[z(-\cos z)+\int \cos z \, dz\right]$$

$$=\frac{8}{\pi}\Big[-z \cos z+\sin z\Big]_0^{\pi/2}=\frac{8}{\pi}.$$

(d) $\displaystyle\int_0^a x \sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)} \, dx=\left(\frac{\pi}{4}-\frac{1}{2}\right) a^2.$

Put $x^2=a^2 \cos 2\theta$ and adjust the limits for θ.

Ex. 13. (a) $\displaystyle\int_0^{\pi/2} \tan x \log \sin x \, dx=-\frac{\pi^2}{24}.$

(b) $\displaystyle\int_0^{\pi/2} \sin x \log \sin x \, dx=\log \left(\frac{2}{e}\right).$

(c) $\displaystyle\int_{-\pi/4}^{\pi/4} \log (\sin x+\cos x) \, dx=\frac{\pi}{4} \log \frac{1}{2},$

(a) $\displaystyle I=\int_0^{\pi/2} \tan x \log \sin x \, dx$

$$=\int_0^{\pi/2} \frac{\sin x}{\cos x} \log \sqrt{(1-\cos^2 x)} \, dx.$$

Now put $\cos x=t$ and adjust the limits for t.

$$I=-\tfrac{1}{2} \int_1^0 \frac{1}{t} \log (1-t^2) \, dt, \quad \because \quad \log \sqrt{x}=\tfrac{1}{2} \log x$$

$$=\tfrac{1}{2} \int_1^0 \frac{1}{t} \left(t^2+\tfrac{1}{2} t^4+\tfrac{1}{3} t^6+\tfrac{1}{4} t^8 \ldots\right) dt$$

$$=\tfrac{1}{2} \int_1^0 \left(t+\tfrac{1}{2} t^3+\tfrac{1}{3} t^5+\tfrac{1}{4} t^7+\ldots\right) dt.$$

Integrating and putting the limits, we get

$$I=-\tfrac{1}{2}\left[\frac{1}{2}+\frac{1}{2.4}+\frac{1}{3.6}+\frac{1}{4.8}+\ldots\right]$$

$$=-\tfrac{1}{2}\left[(1-\tfrac{1}{2})+\tfrac{1}{2}(\tfrac{1}{2}-\tfrac{1}{4})+\tfrac{1}{3}(\tfrac{1}{3}-\tfrac{1}{6})+\tfrac{1}{4}(\tfrac{1}{4}-\tfrac{1}{8})+\ldots\right]$$

$$=-\tfrac{1}{2}\left[\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right\}-\tfrac{1}{2}\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right\}\right]$$

$$=-\tfrac{1}{2}\left(\frac{\pi^2}{6}-\frac{1}{2}\cdot\frac{\pi^2}{6}\right)=-\frac{\pi^2}{24}.$$

It is taken for granted from trigonometry that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots=\frac{\pi^2}{6}.$$

(b) $I=\int_0^{\pi/2} \sin x \log \sin x \, dx.$

Put $\cos x=t$, and adjust the limits.

$$I=-\int_1^0 \log \sqrt{(1-t^2)}\, dt=-\tfrac{1}{2}\int_1^0 -\left(t^2+\frac{t^4}{2}+\frac{t^6}{3}+\ldots\right) dt$$

$$=\tfrac{1}{2}\left(\frac{t^3}{3}+\frac{t^5}{2.5}+\frac{t^7}{3.7}+\ldots\right)_1^0=-\tfrac{1}{2}\left(\frac{1}{3}+\frac{1}{2.5}+\frac{1}{3.7}+\ldots\right)$$

$$=-\left(\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\ldots\right)$$

$$=-\{(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{4}-\tfrac{1}{5})+(\tfrac{1}{6}-\tfrac{1}{7})+\ldots\}$$

$$=-\{(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{4}-\tfrac{1}{5})+(\tfrac{1}{6}-\tfrac{1}{7})+\ldots\}$$

$$=-1+1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\ldots=-1+\log(1+1)$$

$$=\log 2-\log e=\log(2/e), \quad \because \log e=1.$$

(c) $I=\int_{-\pi/4}^{\pi/4} \log(\sin x+\cos x)\, dx$

$$=\int_{-\pi/4}^{\pi/4} \log \sqrt{2} \sin(x+\tfrac{1}{4}\pi)\, dx.$$

Now put $x+\pi/4=t$ and adjust the limits.

$I=\int_0^{\pi/2} \log \sin t+\int_0^{\pi/2} \tfrac{1}{2}\log 2\, dt$ etc. (see Q. 1 P. 4)

Ex. 14. *Prove without performing integration that*

$$\int_{-a}^{2a} \frac{x\,dx}{x^2+p^2}=\int_a^{2a} \frac{x\,dx}{x^2+p^2}.$$

$$\text{L.H.S.}=\int_{-a}^{0} \frac{x\,dx}{x^2+p^2}+\int_0^{2a} \frac{x\,dx}{x^2+p^2}. \qquad \text{(Prop. 3)}$$

Put $x=-t$ in the 1st integral and adjust the limits.

$$\text{L.H S.}=\int_a^0 \frac{t\,dt}{t^2+p^2}+\int_0^{2a} \frac{x\,dx}{x^2+p^2}$$

$$=\int_a^0 \frac{x\,dx}{x^2+p^2}\quad \text{(Prop. 1)}+\int_0^{2a} \frac{x\,dx}{x^2+p^2}$$

$$=\int_a^{2a} \frac{x\,dx}{x^2+p^2}\quad \text{(Prop. 3)}.$$

Ex. 15. (a) $\int_0^{\pi} \sin^3\theta\,(1+2\cos\theta)\,(1+\cos\theta)^2\,d\theta.$

$$I=\int_0^{\pi}\sin^3\theta\,(1+2\cos\theta)\,(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$=\int_0^{\pi}\sin^3\theta\,(1+4\cos\theta+5\cos^2\theta+2\cos^3\theta)\,d\theta$$

$$=2\int_0^{\pi/2}(\sin^3\theta+5\sin^3\theta\cos^2\theta)\,d\theta$$

(See Note Prop. 6 P. 4)

$$=2\left[\frac{2}{3}+5.\frac{2.1}{5.3.1}\right]=\frac{8}{3}.$$ **Ans.**

(b) $\int_0^{\pi} \sin^6 x\,(1-\cos x)^3=\frac{32}{21}.$

(c) $\int_0^{\pi} \sin^6 x\cos^4 x\,dx=\frac{3\pi}{256}.$

Some Typical Solved Examples

Ex. 1. *Prove that* $\int_0^1 \cot^{-1}(1-x+x^2)\,dx=\frac{\pi}{2}-\log 2.$

$$I=\int_0^1 \tan^{-1}\frac{1}{1-x(1-x)}=\int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$$

$$=\int_0^1 \tan^{-1}x\,dx+\int_0^1 \tan^{-1}(1-x)\,dx.$$

$$\because\quad \tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}.$$

Again by prop. IV, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx.$

$$I=\int_0^1 \tan^{-1} x\,dx+\int_0^1 \tan^{-1}\{1-(1-x)\}\,dx$$

$$=\int_0^1 \tan^{-1} x\,dx+\int_0^1 \tan^{-1} x\,dx=2\int_0^1 \tan^{-1} x\,dx$$

$$I=2\left[\left(x\tan^{-1} x\right)_0^1-\int_0^1 \frac{x}{1+x^2}\,dx\right]$$

$$=2\left[\frac{\pi}{4}-\left\{\frac{1}{2}\log(1+x^2)\right\}_0^1\right]=\frac{\pi}{2}-\log 2. \quad \text{Proved.}$$

Ex. 2. *Evaluate* $\int_0^a \sqrt{(a^2-x^2)}\cos^{-1}\frac{x}{a}\,dx.$ (Agra 54)

Put $x=a\cos\theta$ $\therefore$ $dx=-a\sin\theta\,d\theta$ and adjust the limits.

$$\therefore\quad I=\int_{\pi/2}^0 (a\sin\theta).\theta(-a\sin\theta)\,d\theta$$

$$=a^2\int_0^{\pi/2}.\theta\frac{(1-\cos 2\theta)}{2}\,d\theta$$

$$=\frac{a^2}{2}\left[\left(\frac{\theta^2}{2}\right)_0^{\pi/2}-\left\{\left(\theta\frac{\sin 2\theta}{2}\right)_0^{\pi/2}-\frac{1}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta\right\}\right]$$

$$=\frac{a^2}{2}\left[\frac{\pi^2}{8}-0+\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi/2}\right]=\frac{\pi^2a^2}{16}+\frac{a^2}{4}\left(\frac{1}{2}+\frac{1}{2}\right)$$

$$=\frac{\pi^2a^2}{16}+\frac{a^2}{4}.$$

Ex. 3. *Prove that* $\int_{a-\sqrt{(a^2-b^2)}}^{a+\sqrt{(a^2-b^2)}}\frac{(y^2+b^2)\,y\,dy}{\{4a^2y^2-(y^2+b^2)^2\}^{1/2}}=\pi a^2$

Put $(y^2+b^2)=t.$ $\therefore$ $2y\,dy=dt.$

Also when

$y=a+\sqrt{(a^2-b^2)}$ i.e. $y^2=a^2+a^2-b^2+2a\sqrt{(a^2-b^2)}.$

$\therefore$ $t=y^2+b^2=2a\,[a+\sqrt{(a^2-b^2)}]$ and similarly other limit is adjusted to $2a\,[a-\sqrt{(a^2-b^2)}].$

$$\therefore\quad I=\frac{1}{2}\int_{2a[a-\sqrt{(a^2-b^2)}]}^{2a[a+\sqrt{(a^2-b^2)}]}\frac{t\,dt}{[4a^2(t-b^2)-t^2]^{1/2}}.$$

$$=-\frac{1}{4}\int\frac{-2t}{(4a^2t-4a^2b^2-t^2)^{1/2}}\,dt$$

$$=-\frac{1}{4}\left[\int\frac{(-2t+4a^2)\,dt}{(4a^2t-4a^2b^2-t^2)^{1/2}}\right.$$

$$\left.-\frac{4a^2\,dt}{\{4a^2\,(a^2-b^2)-(t-2a^2)^2\}^{1/2}}\right]$$

$$=\left[-\frac{1}{4}.2\,(4a^2t-4a^2b^2-t^2)^{1/2}\right.$$

$$\left.+a^2\sin^{-1}\frac{t-2a^2}{2a\sqrt{(a^2-b^2)}}\right]_{2a\,[a-\sqrt{(a^2-b^2)}]}^{2a\,[(a+\sqrt{(a^2-b^2)}]}.$$

$$(0+a^2\sin^{-1}1)-\{0+a^2\sin^{-1}(-1)\}=a^2.\frac{\pi}{2}+a^2.\frac{\pi}{2}=\pi a^2.$$

Ex. 4. *Prove that* $\int_0^{2\pi} e^{\cos\theta}.\cos(\sin\theta)\,d\theta=2\pi$.

$I=$Real part of $\int_0^{2\pi} e^{\cos\theta}.e^{i\sin\theta}\,d\theta$

$$\because\quad e^{ix}=\cos x+i\sin x$$

$$=\text{Real part of }\int_0^{2\pi}e^{\cos\theta+i\sin\theta}\,d\theta$$

$$=\text{Real part of }\int_0^{2\pi}e^{i\theta}\,d\theta$$

$$=\text{Real part of }\int_0^{2\pi}\left(1+e^{i\theta}+\frac{e^{2i\theta}}{2\,!}+\frac{e^{3i\theta}}{3\,!}+\ldots\right)d\theta$$

$$=\int_0^{2\pi}\left(1+\cos\theta+\frac{1}{2\,!}\cos 2\theta+\frac{1}{3\,!}\cos 3\theta+\ldots\right)$$

$$=\left[\theta+\sin\theta+\frac{\sin 2\theta}{2.(2\,!)}+\frac{\sin 3\theta}{3\,(3\,!)}+\ldots\right]_0^{2\pi}=2\pi.$$

Ex. 4. *Evaluate* $\int_a^{\pi/2}\frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$ *in the form of a series when* $k<1$.

$$I=\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\,d\theta$$

$$=\int_0^{\pi/2}\left[1+\frac{1}{2}k^2\sin^2\theta+\frac{\frac{1}{2}(\frac{1}{2}+1)}{1.2}(k^2\sin^2\theta)^2+\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{1.2.3}(k^2\sin^2\theta)^3+\ldots\right]$$

$$=\left[\theta\right]_0^{\pi/2}+\frac{1}{2}k^2\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)+\frac{1.3}{2.4}.k^4.\frac{3.1}{4.2}\frac{\pi}{2}+\frac{1.3\ 5}{2.4.6}k^6.\frac{5.3.1}{2.4.6}.\frac{\pi}{2}+\ldots$$

$$=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\ 3}{2.4}\right)^2k^4+\left(\frac{1\ 3.5}{2.4.6}\right)^2k^6+\ldots\right].$$

Note 2. Factors of $1-2a\cos\theta+a^2$.

$$1-2a\cos\theta+a^2=1-a(e^{i\theta}+e^{-i\theta})+ae^{i\theta}.ae^{-i\theta}.$$

$$\therefore\quad 1-2a\cos\theta+a^2=(1-ae^{i\theta})(1-ae^{-i\theta}).$$

Similarly, $1+2a\cos\theta+a^2=(1+ae^{i\theta})\ (1+ae^{-i\theta})$.

Note. 2. $\int_0^{\pi}\cos mx\cos nx\,dx=\int_0^{\pi}\sin mx\sin nx\,dx=0,$

when $m\neq n$ and $=\frac{\pi}{2}$ when $m=n$.

$$I=\frac{1}{2}\int_0^{\pi}[\cos(m+n)x\pm\cos(m-n)x]\,dx$$

$$=\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}\pm\frac{\sin(m-n)x}{m-n}\right]_0^{\pi}=0,\text{ when } m\neq n,$$

when $m=n$, then $I=\frac{1}{2}\int_0^{\pi}(1\pm\cos 2mx)$

$$=\frac{1}{2}\left\{x\pm\frac{\sin 2mx}{2m}\right\}_0^{\pi}$$

Ex. 5. *Find the value of* $\int_0$ *log (1*

according as $a<1$ or >1. (Delhi Hon

Case 1. $a<1$.

$$I=\int_0^{\pi}\log(1-ae^{ix})(1-ae^{-ix})\,dx\qquad\text{by note (1)}$$

or $I=\int_0^\pi [\log (1-ae^{ix})+\log (1-ae^{-ix})]\, dx$

$$=-\int_0^\pi \left[\left(ae^{ix}+\frac{a^2}{2}e^{2ix}+\frac{a^3}{3}e^{3ix}+\ldots\right)+\left(ae^{-ix}+\frac{a^2}{2}e^{-2ix}+\frac{a^3}{3}e^{-3ix}+\ldots\right)\right]$$

$$=-\int_0^\pi \left[a\,(2\cos x)+\frac{a^2}{2}(2\cos 2x)+\frac{a^3}{3}(2\cos 3x)+\ldots\right]$$

$$=-2\left[a\sin x+\frac{a^2}{2}\frac{\sin 2x}{2}+\frac{a^3}{3}\frac{\sin 3x}{3}\right]_0^\pi=0 \because \sin r\pi=0.$$

Case 2. $a > 1$.

$$\int_0^\pi \log (1-2a\cos x+a^2)\, dx=\int_0^\pi \log a^2\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)dx$$

$$=\int_0^\pi \log a^2\, dx+\int_0^\pi \log (1-2k\cos x+k^2)\, dx,$$

where $k=\frac{1}{a}$ and $a > 1$; $\therefore$ $k < 1$.

$\therefore$ $I=\Big[x\log a^2\Big]_0^\pi+0$ by case $1=\pi\log a^2=2\pi\log a$.

Ex. 6. *Integrate* $\int \log (1+2a\cos x+a^2)\, dx$.

Proceed exactly as above.

Case 1. $a < 1$.

$$I=\int [\log (1+ae^{ix})+\log (1+ae^{-ix})]\, dx$$

$$=\int\left[\left(ae^{ix}-\frac{a^2}{2}e^{2ix}+\ldots\right)+\left(ae^{-ix}-\frac{a^2}{2}e^{-2ix}+\ldots\right)\right]dx$$

$$=\int\left[a\,(2\cos x)-\frac{a^2}{2}(2\cos 2x)+\frac{a^3}{3}(2\cos 3x)-\ldots\right]dx$$

$$=2\left[a\sin x-\frac{a^2}{2}\frac{\sin 2x}{2}+\frac{a^3}{3}\frac{\sin 3x}{3}\ldots\right]$$

If $a > 1$, then $I = \int \log a^2 \left(\frac{1}{a^2} + \frac{2}{a} \cos x + 1\right)$

$= \int \log a^2 \, dx + \int \log (1 + 2k \cos x + k^2) \, dx$, where $k = \frac{1}{a} < 1$

$= x \log a^2 + 2 \left(k \sin x - \frac{k^2}{2} \frac{\sin 2x}{2} + \frac{k^3}{3} \frac{\sin 3x}{3} \dots \right)$

[by case 1]

$= 2x \log a + 2 \left(\frac{1}{a} \sin x - \frac{1}{2a^2} \frac{\sin 2x}{2} + \frac{1}{3a^3} \frac{\sin 3x}{3} \dots \right)$.

Ex. 7. *Evaluate* $\int_0^\pi \log (1 - 2a \cos \theta + a^2) \cos n\theta \, d\theta$, $a^2 < 1$.

(Agra 44)

Proceeding exactly as in Ex. 5,

$$I = -2 \int_0^\pi \left(a \cos \theta + \frac{a^2}{2} \cos 2\theta + \frac{a^3}{3} \cos 3\theta + \dots \right) \cos n\theta \, d\theta$$

$$= -2 \int_0^\pi \left(a \cos \theta \cos n\theta + \frac{a^2}{2} \cos 2\theta \cos n\theta + \dots \right.$$

$$\left. \dots + \frac{a^n}{n} \cos n\theta \cos n\theta \right) d\theta$$

$= -2 \left(0 + \frac{a^n}{n} \cdot \frac{\pi}{2}\right)$ [by Note 2 P. 23] $= -\frac{\pi a^n}{n}$.

Ex. 8. *Prove that* $\int_0^\pi \frac{dx}{(1 - 2a \cos x + a^2)} = \frac{\pi}{1 - a^2}$ *or* $\frac{\pi}{a^2 - 1}$

according as $a < 1$ *or* > 1.

Case I. $a < 1$.

$$I = \int_0^\pi \frac{dx}{(1 - ae^{ix})(1 - ae^{-ix})}$$

$$= \frac{1}{1 - a^2} \int_0^\pi \frac{1 - ae^{-ix} + ae^{-ix} - a^2 e^{ix} \cdot e^{-ix}}{(1 - ae^{ix})(1 - ae^{-ix})} dx$$

$$= \frac{1}{(1 - a^2)} \int_0^\pi [(1 - ae^{ix})^{-1} + ae^{-ix} (1 - \dots$$

$$=\frac{1}{(1-a^2)}\int_0^\pi [(1+ae^{ix}+a^2e^{2ix}+a^3e^{3ix}+\ldots$$
$$+ae^{-ix}(1+ae^{-ix}+a^2e^{-2ix}+a^3e^{-3ix}+\ldots)]\,dx$$

$$=\frac{1}{(1-a^2)}\int [1+a(e^{ix}+e^{-ix})+a^2(e^{2ix}+e^{-2ix})+a^3(e^{3ix}+e^{-3ix})$$
$$+\ldots]\,dx$$

$$=\frac{1}{(1-a^2)}\int_0^\pi (1+2a\cos x+2a^2\cos 2x+2a^3\cos 3x+\ldots)\,dx$$

$$=\frac{1}{(1-a^2)}\left[\left(x+2a\sin x+2a^2\frac{\sin 2x}{2}+2a^3\frac{\sin 3x}{3}+\ldots\right)\right]_0^\pi$$

$$=\frac{\pi}{1-a^2}.$$

Case II. $a > 1.$

$$I=\int_0^\pi \frac{dx}{a^2\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)}.\quad \text{Put}\quad k=\frac{1}{a}<1.$$

$$I=\frac{1}{a^2}\int_0^\pi \frac{dx}{(1-2k\cos x+k^2)}=\frac{1}{a^2}\frac{\pi}{1-k^2}=\frac{1}{a^2}\left\{\frac{\pi}{1-(1/a^2)}\right\}$$

$$=\frac{\pi}{a^2-1}.$$

Ex. 9. *Prove that* $\int_0^\pi \frac{\cos rx}{1-2a\cos x+a^2}\,dx=\frac{\pi a^r}{1-a^2}.$

Proceeding exactly as in Ex. 8 case I,

$$I=\frac{1}{(1-a^2)}\int_0^\pi (1+2a\cos x+2a^2\cos 2x+2a^3\cos 3x+\ldots$$
$$\ldots+2a^r\cos rx+\ldots)\cos rx\,dx$$

$$=\frac{1}{(1-a^2)}\left(0+0+0+\ldots+2a^r.\frac{\pi}{2}\right)\quad \text{by note 2 P. 23}$$

$$=\frac{\pi a^r}{1-a^2}.\qquad\qquad \text{Proved}$$

§ 2. Differentiation under the sign of integration.

The method is very important and by its help we can integrate functions which are not easily integrable otherwise. The integrand of definite integral is first differentiated with respect to a quantity of which the limits of integration are independent. It is applied in the followidg two ways :—

I. Suppose we know a certain integral and its value ; then by differentiating both sides we may get a new integral on one side and its value on the other.

II. Suppose certain integrals are given which on differentiation take the form which is easily integrable. In such cases the value of the integral (which occurs after differentiation) is calculated and then it is integrated with respect to the quantity with which the original integral was differentiated.

The application of above method will be more clear by the following solved examples. Students will have no difficulty in acquiring the working rule after reading a few solved examples.

Ex. 1. *Evalnate* $\int_0^\pi \frac{dx}{a+b\cos x}$, $a > b$, *and hence deduce that* $\int_0^\pi \frac{dx}{(a+b\cos x)^2}=\frac{\pi a}{(a^2-b^2)^{3/2}}$. **(Gujrat 58)**

$$I=\int_0^\pi \frac{dx}{a+b\cos x}$$

$$=\int_0^\pi \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}.$$

Divide above and below by $\cos^2\frac{x}{2}$ and put $\tan\frac{x}{2}=t$ and adjust the limits.

$$\therefore\quad I=\int_0^\infty \frac{2\,dt}{(a+b)+(a-b)\,t^2}=\frac{2}{(a-b)}\int_0^\infty \frac{dt}{\left\{\sqrt{\left(\frac{a+b}{a-b}\right)^2}+t^2\right\}}$$

$$=\frac{2}{a-b}\cdot\sqrt{\left(\frac{a-b}{a+b}\right)}\left[\tan^{-1} t\sqrt{\left(\frac{a+b}{a-b}\right)}\right]_0^\infty$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\cdot\left[\frac{\pi}{2}-0\right]=\frac{\pi}{\sqrt{(a^2-b^2)}}.$$

$$\therefore \int_0^\pi \frac{dx}{(a+b\cos x)}=\frac{\pi}{\sqrt{(a^2-b^2)}}.$$

Differentiating both sides w.r.t. a, we get

$$\int_0^\pi -\frac{dx}{(a+b\cos x)^2}=-\frac{\pi\times 2a}{2\,(a^2-b^2)^{3/2}}.$$

$$\therefore \int_0^\pi \frac{dx}{(a+b\cos x)^2}=\frac{\pi a}{(a^2-b^2)^{3/2}}.$$ **Proved.**

Ex. 2. *Prove that* $\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi\,(a^2+b^2)}{4a^3b^3}.$

(Agra 40)

Here we can easily calculate $I=\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)}$ as under.

Divide above and below by $\cos^2 x$.

$$\therefore \quad I=\int_0^{\pi/2}\frac{\sec^2 x\,dx}{b^2+a^2\tan^2 x}=\frac{1}{b}\cdot\frac{1}{a}\left[\tan^{-1}\frac{a\tan x}{b}\right]_0^{\pi/2}=\frac{\pi}{2ab}$$

or $$\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)}=\frac{\pi}{2ab}. \qquad \ldots(1)$$

Now differentiating both sides first w.r.t. a and then w.r.t. b, we get

$$\int_0^{\pi/2}\frac{-2a\sin^2 x}{(a^2\sin^2 x+b^2\cos^2 x)^2}\,dx=\frac{-\pi}{2a^2b}.$$

or $$\int_0^{\pi/2}\frac{\sin^2 x\,dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi}{4a^3b}. \qquad \ldots(2)$$

Again differentiating both sides of (1) w.r.t. b, we have as above

$$\int_0^{\pi/2}\frac{\cos^2 x\,dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi}{4ab^3}. \qquad \ldots(3)$$

Adding (2) and (3) and putting $\sin^2 x+\cos^2 x=1$, we get

$$\int_0^{\pi/2}\frac{dx}{(a^2\sin^2 x+b^2\cos^2 x)^2}=\frac{\pi}{4a^3b}+\frac{\pi}{4ab^3}=\frac{\pi(a^2+b^2)}{4a^3b^3}.$$

Ex. 3. (a) *Prove that* $\int_0^{\infty} x^n e^{-ax}\,dx=\frac{n\,!}{a^{n+1}}$.

We know $\int_0 e^{-ax}\,dx=-\frac{1}{a}\left[e^{-ax}\right]_0^{\infty}=\frac{-1}{a}[0-1]=\frac{1}{a}$.

Differentiating both sides n times w.r.t. a,

$$\int_0^{\infty}(-x)^n e^{-ax}\,dx=\frac{(-1)^n\,n\,!}{a^{n+1}}.$$

$$\therefore\quad \int_0^{\infty} x^n e^{-ax}\,dx=\frac{n\,!}{a^{n+1}}.$$

(b) *Prove* $\int_0^{\infty} e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ *and hence deduce that* $\int_0^{\infty} e^{-ax}.x^n\sin bx=\frac{n\,!\,\sin(n+1)\,\theta}{(a^2+b^2)^{(n+1)/2}}$, *where* $\theta=\tan^{-1}\frac{b}{a}$.

Integrating by parts, we get

$$I=-\frac{e^{-ax}}{a}\sin bx+\frac{b}{a}\int e^{-ax}\cos bx\,dx$$

$$=-\frac{e^{-ax}}{a}\sin bx+\frac{b}{a}\left[-\frac{e^{-ax}}{a}\cos bx-\frac{b}{a}\int e^{-ax}\sin bx\,dx\right].$$

$$\therefore\quad I\left(1+\frac{b^2}{a^2}\right)=-\frac{e^{-ax}}{a}\sin bx-\frac{b}{a^2}e^{-ax}\cos bx.$$

$$\therefore\quad I=\int_0^{\infty} e^{-ax}\sin bx\,dx=-\left[\frac{e^{-ax}}{a^2+b^2}(a\sin bx+b\cos bx)\right]_0^{\infty}$$

$$=-\left[0-\frac{b}{a^2+b^2}\right]=\frac{b}{a^2+b^2}.$$

Now differentiate both sides n times w.r.t. a (Note) and remember that

$$\frac{d^n}{dx^n}\left(\frac{1}{x^2+k^2}\right)=\frac{(-1)^n\,n\,!\,\sin(n+1)\,\theta\,\sin^{n+1}\theta}{k^{n+2}}.$$

where $\theta=\tan^{-1}k/x$.

(Diff. Cal for B.Sc. P. 67)

$$\therefore \int e^{-ax}(-x)^n \sin bx\, dx = \frac{b\,(-1)^n\,(n\,!)\sin(n+1)\,\theta \sin^{n+1}\theta}{b^{n+2}},$$

where $\theta = \tan^{-1} b/a$.

Now $\tan\theta = \dfrac{b}{a}$; $\therefore$ $\sin\theta = \dfrac{b}{\sqrt{(a^2+b^2)}}$.

$$\therefore \int_0^\infty e^{-ax}.x^n \sin bx\, dx = \frac{b.(n\,!).\sin(n+1)\,\theta}{b^{n+2}} \cdot \frac{b^{n+1}}{(a^2+b^2)^{(n+1)/2}}$$

or $\displaystyle\int_0^\infty e^{-ax}\, x^n \sin bx\, dx = \frac{n\,!\sin(n+1)\,\theta}{(a^2+b^2)^{(n+1)/2}}$, where $\tan\theta = \dfrac{b}{a}$.

(c) *Evaluate* $\displaystyle\int_0^\infty e^{-ax} \cos bx\; dx = \frac{a}{a^2+b^2}$ *and hence deduce that* $\displaystyle\int_0^\infty e^{-ax} x^n \cos bx\; dx = \frac{(n\,!)\cos(n+1)\,\theta}{(a^2+b^2)^{(n+1)/2}}$, *where* $\theta = \tan^{-1}\dfrac{b}{a}$.

Integrating by parts, we get

$$I = -\frac{e^{-ax}}{a}\cos bx - \frac{b}{x}\int e^{-ax}\sin bx\, dx$$

$$= -\frac{e^{-ax}}{a}\cos bx - \frac{b}{a}\left[-\frac{e^{-ax}}{a}\sin bx + \frac{b}{a}\int e^{-ax}\cos bx\, dx\right]$$

or
$$I\left(1+\frac{b^2}{a^2}\right) = -\frac{e^{-ax}}{a}\cos bx + \frac{b}{a^2}\, e^{-ax}\sin bx.$$

$$\therefore\quad I = \int_0^\infty e^{-ax}\cos bx\, dx = -\left[\frac{e^{-ax}}{(a^2+b^2)}(a\cos bx - b\sin bx)\right]_0^\infty$$

$$= -\left[0 - \frac{a}{a^2+b^2}\right] = \frac{a}{a^2+b^2}.$$

Now differentiate both sides n times w.r.t. a and remember that

$$\frac{d^n}{dx^n}\frac{x}{(x^2+k^2)} = \frac{(-1)^n\, n\,!}{k^{n+1}}\cos(n+1)\,\theta\,\sin^{n+1}\theta,$$

where $\theta = \tan^{-1} k/x$.

(Diff. Cal. for B.Sc. Page 73)

$\therefore \quad \int_0^\infty e^{-ax} (-x)^n \cos bx \, dx = \frac{(-1)^n \, n \, ! \cos (n+1) \, \theta \, \sin^{n+1}\theta}{b^{n+1}}$

where $\theta = \tan^{-1} \frac{b}{a}$

or $\quad \int_0^\infty e^{-ax} x^n \cos bx \, dx = \frac{n \, ! \cos (n+1) \, \theta}{b^{n+1}} \cdot \frac{b^{n+1}}{(a^2+b^2)^{(n+1)/2}}$

$= \frac{n \, ! \cos (n+1) \, \theta}{(a^2+b^2)^{(n+1)/2}}$, where $\tan \theta = \frac{b}{a}$.

Rule 1. *From the above five examples it is clear that we knew a certain integral (or whose value we could find easily). Then on differentiating both sides with respect to a certain quantity, we get a new integral in L.H.S. and its value on the R.H.S. This is one way of using the method of differentiation under the sign of integration. Another method will be given after examples 4 and 5.*

Ex. 4. *Evaluate* $\int_0^\infty \frac{\log (1+a^2x^2)}{1+b^2x^2} dx.$ **(Rajputana 63 ; Delhi Hons. 57, 60 ; Vikram 63 ; Agra 64, 54, 49 ; Pb. 60)**

$$I = \int_0^\infty \frac{\log (1+a^2x^2)}{1+b^2x^2} \, dx.$$

Differentiating w.r.t. a, we get

$$\frac{dI}{da} = \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(1+b^2x^2)}.$$

Split into partial fractions by method of suppression, *i.e.* by putting $x^2 = -\frac{1}{a^2}$ and then $x^2 = -\frac{1}{b^2}$, we get

$$\frac{dI}{da} = 2a \int_0^\infty \frac{1}{(b^2-a^2)} \left\{ \frac{1}{1+a^2x^2} - \frac{1}{1+b^2x^2} \right\} dx$$

$$= \frac{2a}{b^2-a^2} \cdot \left[\frac{1}{a} \tan^{-1} ax - \frac{1}{b} \tan^{-1} bx \right]_0^\infty$$

or $\quad \frac{dI}{da} = \frac{2a}{b^2-a^2} \left[\frac{1}{a} \cdot \frac{\pi}{2} - \frac{1}{b} \cdot \frac{\pi}{2} \right] = \frac{\pi}{b\,(a+b)}.$

Now integrating both sides w. r. t. a the quan[illegible]
which we differentiated the original integral,

$$I=\int \frac{\pi}{b}\frac{da}{(a+b)}+c=\frac{\pi}{b}\log(a+b)+c$$

In order to find the value of c, we use th[illegible]
when $a=0$, then $I=0$ as $\log 1=0$. Hence putting
$I=0$ in (1), we get

$$0=\frac{\pi}{b}\log b+c\ ;\quad \therefore\ c=-\frac{\pi}{b}\log b.$$

$$\therefore\ I=\frac{\pi}{b}[\log(a+b)-\log b]=\frac{\pi}{b}\log\frac{a+b}{b}.$$

$$\therefore\ \int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx=\frac{\pi}{b}\log\frac{a+b}{b},\ a>b$$

Ex. 5. *Evaluate* $\int_0^{\pi/2} \log\frac{(a+b\sin\theta)}{(a-b\sin\theta)}\frac{d\theta}{\sin\theta}$.

(Delhi Hons. 58, 61;

Let the given integral be denoted by I.

$$I=\int_0^{\pi/2} [\log(a+b\sin\theta)-\log(a-b\sin\theta)]$$

Differentiating both sides w.r t. b, we g[illegible]

$$\frac{dI}{db}=\int_0^{\pi/2}\left[\frac{\sin\theta}{a+b\sin\theta}-\frac{-\sin}{a-}\right.$$

or

$$\frac{dI}{db}=\int_0^{\pi/2}\frac{2a}{a^2-b^2}$$

Divide above and below by
$1+\tan^2\theta$ in D^r

$$\frac{dI}{db}=\int_0^{\pi/2}\frac{2a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)-b^2\,\mathrm{ta}}$$

Divide above and below by
$t\tan x=t$.

(Partially visible left page fragment:) $\therefore \int e^{-ax}$ (where $\theta=$ Now $\therefore \int_0^\infty e^{-ax}$ or $\int_0^\infty e^{-ax}$. (c) *Eva* *that* $\int_0^\infty e^{-ax}x$ Integrati $I=-\frac{e^{-ax}}{a}$ $=-\frac{e^{-ax}}{a}$ or $I(1$ $\therefore I=\int_0^\infty e^{-a}$ Now differ member that $\frac{d^n}{dx^n}\frac{x}{(x^2+k}$

Put $\sqrt{(a^2-b^2)}\tan\theta=t$; $\therefore$ $\sqrt{(a^2-b^2)}\sec^2\theta\,d\theta=dt$ and adjust the limits.

$$\therefore \quad \frac{dI}{db}=\frac{2a}{\sqrt{(a^2-b^2)}}\int_0^\infty \frac{dt}{a^2+t^2}=\frac{2a}{\sqrt{(a^2-b^2)}}\cdot\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^\infty$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{(a^2-b^2)}}.$$

Now integrate both sides w.r.t b the quantity w.r.t. which we differentiated the original integral

and $$\int\frac{dx}{\sqrt{(a^2-x^2)}}=\sin^{-1}\frac{x}{a}.$$

$$\therefore \quad I=\pi\int\frac{db}{\sqrt{(a^2-b^2)}}+c=\pi\sin^{-1}\frac{b}{a}+c.$$

But when $b=0$, $I=0$ $\because$ $\log I=0$. Also $\sin^{-1}0=0$.

$$\therefore \quad 0=0+c\,; \quad \therefore \quad c=0.$$

$$\therefore \quad I=\int_0^{\pi/2}\log\frac{(a+b\sin\theta)}{(a-b\sin\theta)}\frac{d\theta}{\sin\theta}=\pi\sin^{-1}\frac{b}{a}.$$

Rule 2. *In examples no. 4 and 5 we observed that on differentiating the given integral w.r.t. a certain quantity it took the form which could be easily integrated. After having calculated the value of the integral (obtained after differentiating the original integral) we integrate w. r. t. the quantity with which the original integral was differentiated.*

Below we shall give some more examples to illustrate the above two ways of integrating by the principle of differentiation under the sign of integration.

Ex. 6. *Evaluate* $\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta.$

(Delhi Hons. 59, 55 ; Agra 61, 57, 36)

$$I=\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta.$$

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^{\pi/2}\frac{2\alpha\cos^2\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}\,d\theta=\int_0^{\pi/2}\frac{2\alpha\,d\theta}{\alpha^2+\beta^2\tan^2\theta}$$

Now integrating both sides w. r. t. a the quantity w. r. t. which we differentiated the original integral,

$$I=\int \frac{\pi\, da}{b\,(a+b)}+c=\frac{\pi}{b}\log(a+b)+c. \qquad \ldots(1)$$

In order to find the value of c, we use the fact that when $a=0$, then $I=0$ as $\log 1=0$. Hence putting $a=0$ and $I=0$ in (1), we get

$$0=\frac{\pi}{b}\log b+c\,; \quad \therefore \quad c=-\frac{\pi}{b}\log b.$$

$$\therefore \quad I=\frac{\pi}{b}\left[\log(a+b)-\log b\right]=\frac{\pi}{b}\log\frac{a+b}{b}.$$

$$\therefore \quad \int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx=\frac{\pi}{b}\log\frac{a+b}{b},\ a>b.$$

Ex. 5. *Evaluate* $\displaystyle\int_0^{\pi/2} \log\frac{(a+b\sin\theta)}{(a-b\sin\theta)}\frac{d\theta}{\sin\theta}$.

(Delhi Hons. 58, 62 ; Agra 43, 58)

Let the given integral be denoted by I.

$$I=\int_0^{\pi/2}\left[\log(a+b\sin\theta)-\log(a-b\sin\theta)\right]\frac{d\theta}{\sin\theta}.$$

Differentiating both sides w.r.t. b, we get

$$\frac{dI}{db}=\int_0^{\pi/2}\left[\frac{\sin\theta}{a+b\sin\theta}-\frac{-\sin\theta}{a-b\sin\theta}\right]\frac{d\theta}{\sin\theta}$$

or

$$\frac{dI}{db}=\int_0^{\pi/2}\frac{2a}{a^2-b^2\sin^2\theta}\,d\theta.$$

Divide above and below by $\cos^2\theta$ and replace $\sec^2\theta$ by $1+\tan^2\theta$ in D^r

$$\frac{dI}{db}=\int_0^{\pi/2}\frac{2a\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)-b^2\tan^2\theta}=\int_0^{\pi/2}\frac{2a\sec^2\theta}{a^2+\{\sqrt{(a^2-b^2)}\tan\theta\}^2}.$$

$$\int\frac{dx}{a^2-b^2\sin^2 x}.$$

Divide above and below by $\cos^2 x$ and change $\sec^2 x$ in D^r to $1+\tan^2 x$ and then put $\tan x=t$.

Put $\sqrt{(a^2-b^2)}\tan\theta=t$; $\therefore$ $\sqrt{(a^2-b^2)}\sec^2\theta\, d\theta=dt$ and adjust the limits.

$$\therefore \quad \frac{dI}{db}=\frac{2a}{\sqrt{(a^2-b^2)}}\int_0^\infty \frac{dt}{a^2+t^2}=\frac{2a}{\sqrt{(a^2-b^2)}}\cdot\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^\infty$$

$$=\frac{2}{\sqrt{(a^2-b^2)}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{(a^2-b^2)}}.$$

Now integrate both sides w.r.t. b the quantity w.r.t. which we differentiated the original integral

and
$$\int\frac{dx}{\sqrt{(a^2-x^2)}}=\sin^{-1}\frac{x}{a}.$$

$$\therefore \quad I=\pi\int\frac{db}{\sqrt{(a^2-b^2)}}+c=\pi\sin^{-1}\frac{b}{a}+c.$$

But when $b=0$, $I=0$ $\because$ $\log 1=0$. Also $\sin^{-1}0=0$.

$$\therefore \quad 0=0+c\ ; \quad \therefore \quad c=0.$$

$$\therefore \quad I=\int_0^{\pi/2}\log\frac{(a+b\sin\theta)}{(a-b\sin\theta)}\,\frac{d\theta}{\sin\theta}=\pi\sin^{-1}\frac{b}{a}.$$

Rule 2. *In examples no. 4 and 5 we observed that on differentiating the given integral w.r.t. a certain quantity it took the form which could be easily integrated. After having calculated the value of the integral (obtained after differentiating the original integral) we integrate w. r. t. the quantity with which the original integral was differentiated.*

Below we shall give some more examples to illustrate the above two ways of integrating by the principle of differentiation under the sign of integration.

Ex. 6. *Evaluate* $\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta.$

(Delhi Hons. 59, 55 ; Agra 61, 57, 36)

$$I=\int_0^{\pi/2}\log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta.$$

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^{\pi/2}\frac{2\alpha\cos^2\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}\,d\theta=\int_0^{\pi/2}\frac{2\alpha\, d\theta}{\alpha^2+\beta^2\tan^2\theta}$$

$$=\int_0^{\pi/2} \frac{2\alpha \sec^2\theta \, d\theta}{(1+\tan^2\theta)(\alpha^2+\beta^2\tan^2\theta)}.$$

Put $\tan\theta=t$ and adjust the limits.

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^{\infty} \frac{2\alpha \, dt}{(1+t^2)(\alpha^2+\beta^2 t^2)}.$$

Split into partial fractions by method of suppression by putting $t^2=-1$ and $-\dfrac{\alpha^2}{\beta^2}$.

$$\therefore \quad \frac{dI}{d\alpha}=\frac{2\alpha}{(\beta^2-\alpha^2)}\int_0^{\infty}\left\{\frac{\beta^2}{(\alpha^2+\beta^2t^2)}-\frac{1}{1+t^2}\right\} dt$$

$$=\frac{2\alpha}{\beta^2-\alpha^2}\left[\frac{\beta}{\alpha}\tan^{-1}\frac{\beta t}{\alpha}-\tan^{-1} t\right]_0^{\infty}$$

$$=\frac{2\alpha}{\beta^2-\alpha^2}\left[\frac{\beta}{\alpha}-1\right].\frac{\pi}{2}=\frac{\pi}{(\alpha+\beta)}.$$

Now integrating w.r.t. α the quantity w.r.t. which we differentiated the original integral,

$$I=\pi \log(\alpha+\beta)+c.$$

But when $\beta=\alpha$, then

$$I=\int_0^{\pi/2}\log\alpha^2.1\,d\theta=\left[\theta\log\alpha^2\right]_0^{\pi/2}=\frac{\pi}{2}.2\log\alpha=\pi\log\alpha.$$

$$\therefore \quad \pi\log\alpha=\pi\log 2\alpha+c.$$

$$\therefore \quad c=\pi(\log\alpha-\log 2\alpha)=\pi\log\tfrac{1}{2}.$$

$$\therefore \quad I=\pi\log(\alpha+\beta)+\pi\log\tfrac{1}{2}=\pi\log\frac{\alpha+\beta}{2}.$$

Ex. 7. *Evaluate* $\displaystyle\int_0^{\pi/2}\frac{\log(1+\cos\alpha.\cos x)}{\cos x}\,dx.$

(Agra 63, 56, 48)

$$I=\int_0^{\pi/2}\frac{\log(1+\cos\alpha.\cos x)}{\cos x}\,dx.$$

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^{\pi/2}\frac{1}{\cos x\,(1+\cos\alpha\cos x)}(-\sin\alpha.\cos x)\,dx$$

$$=\int_0^{\pi/2} \frac{-\sin \alpha . dx}{\left(\cos^2 \frac{x}{2}+\sin^2 \frac{x}{2}\right)+\cos \alpha \left(\cos^2 \frac{x}{2}-\sin^2 \frac{x}{2}\right)}.$$

Divide above and below by $\cos^2 \frac{x}{2}$ and put $\tan \frac{x}{2}=t$ and adjust the limits.

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^1 \frac{-\sin \alpha .(2\, dt)}{(1+t^2)+\cos \alpha\, (1-t^2)}$$

$$=\int_0^1 \frac{-2 \sin \alpha\, dt}{(1+\cos \alpha)+(1-\cos \alpha).t^2}$$

$$=\frac{-2 \sin \alpha}{1-\cos \alpha} \int_0^1 \frac{dt}{\dfrac{2 \cos^2 \alpha/2}{2 \sin^2 \alpha/2}+t^2}$$

$$=-\frac{4 \sin \alpha/2 \cos \alpha/2}{2 \sin^2 \alpha/2} \cdot \frac{1}{\cot \alpha/2}\left[\tan^{-1} \frac{t}{\cot \alpha/2}\right]_0^1.$$

$$\therefore \quad \frac{dI}{d\alpha}=-2 . \tan^{-1}\left(\tan \frac{\alpha}{2}\right)=-2 . \frac{\alpha}{2}=-\alpha.$$

Integrating w.r.t. α the quantity w.r.t. which we differentiated the original integral,

$$I=-\tfrac{1}{2}\alpha^2+c^2.$$

But when $\alpha=\frac{\pi}{2}$, $\cos \alpha=0$ and $\log 1=0$; $\therefore \ I=0.$

$$\therefore \quad 0=-\frac{1}{2} . \frac{\pi^2}{4}+c \ ; \quad \therefore \quad c=\frac{\pi^2}{8}.$$

$$\therefore \quad I=\frac{\pi^2}{8}-\frac{1}{2}\alpha^2=\frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right).$$

Ex. 8. (a) *Show that if*

$$-1 < a < 1 \text{ and } -\frac{\pi}{2} < \sin^{-1} a < \frac{\pi}{2},$$

$$\int_0^\pi \frac{\log (1+a \cos x)}{\cos x}=\pi \sin^{-1} a. \qquad \textbf{(Gujrat 52)}$$

Proceeding exactly as in Q. 7, we get

$$\frac{dI}{da}=\int_0^\infty \frac{2\,dt}{(1+a)+(1-a)\,t^2}, \text{ we have put } 1-a \text{ as } a<1 \text{ given}$$

$$=\frac{2}{1-a}\int_0^\infty \frac{dt}{\left\{\sqrt{\left(\frac{1+a}{1-a}\right)}\right\}^2+t^2}$$

$$=\frac{2}{1-a}\sqrt{\left(\frac{1-a}{1+a}\right)}\left[\tan^{-1} t\sqrt{\left(\frac{1-a}{1+a}\right)}\right]_0^\infty$$

or $$\frac{dI}{da}=\frac{2}{\sqrt{(1-a^2)}}\cdot\frac{\pi}{2}=\frac{\pi}{\sqrt{(1-a^2)}}.$$

Integrating, we get

$$I=\pi\sin^{-1} a+c.$$

When $a=0$, $I=0$; $\therefore$ $c=0$ Hence $I=\pi\sin^{-1} a$

(b) *Prove that* $\displaystyle\int_0^\pi \frac{\log(1+\sin\alpha\cos x)}{\cos x}\,dx=\alpha\pi.$

It is same as part (a). Replace $\sin\alpha$ by a, so that

$$I=\pi\sin^{-1} a \text{ by part (a)}$$

or $$I=\pi\sin^{-1}(\sin\alpha)=\pi\alpha.$$

Ex. 9. *Evaluate* $\displaystyle\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx.$

(Vikram 62 ; Delhi Hons. 54 ; Burdwan Hons. 64 ; Rajputana 52 ; Nagpur 60 ; Agra 56)

$$I=\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}.$$

Differentiating w.r.t. a, we get

$$\frac{dI}{da}=\int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{x}{1+a^2x^2}\,dx=\int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\,dx.$$

Splitting into fractions by method of suppression,

$$\frac{dI}{da}=\int_0^\infty\left\{\frac{1}{(1-a^2)(1+x^2)}-\frac{a^2}{(1-a^2)(1+a^2x^2)}\right\}dx$$

or $$\frac{dI}{da}=\frac{1}{1-a^2}\left[\tan^{-1} x-\frac{a^2}{a}\tan^{-1} ax\right]_0^\infty$$

$$=\frac{1}{1-a^2}\left[\frac{\pi}{2}-a\,\frac{\pi}{2}\right]=\frac{\pi}{2(1+a)}.$$

Integrating both sides w.r.t a, *we get*

$$I=\frac{\pi}{2}\log(1+a)+c.$$

When $a=0$, $I=0$. $\therefore$ $c=0$ as $\log 1=0$.

$$\therefore \quad I=\frac{\pi}{2}\log(1+a).$$

Ex. 10. *Prove that* $\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx=\tan^{-1}\frac{b}{a}$.

Hence deduce that $\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2}$.

(Poona 60 ; Nagpur 63)

$$I=\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx.$$

Differentiate both sides w.r.t. b,

$$\frac{dI}{db}=\int_0^\infty \frac{e^{-ax}\cos bx}{x}.x\,dx=\int_0^\infty e^{-ax}\cos bx\,dx$$

or $$\frac{dI}{db}=\frac{a}{a^2+b^2}.$$ [Refer Ex. 3 (c) P. 30]

Integrating both sides w.r.t. b, we get

$$I=a.\frac{1}{a}\tan^{-1}\frac{b}{a}+c=\tan^{-1}\frac{b}{a}+c.$$

When $b=0$, $I=0$; $\therefore$ $c=0$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx=\tan^{-1}\frac{b}{a}.$$

Again putting $a=0$ and $\tan^{-1}\infty=\frac{\pi}{2}$ in the above, we get

$$\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2}.$$

The student would have thought that why not calculate $\frac{dI}{da}$ in this question instead of $\frac{dI}{db}$ as it will also eliminate x

i.e., $\frac{dI}{da}=\int_0^\infty \frac{e^{-ax}\sin bx}{x}(-x)\,dx=-\int_0^\infty e^{-ax}\sin bx\,dx$

or $\frac{dI}{da}=-\frac{b}{a^2+b^2}$. [Refer Ex. 3 (b) P. 29]

Integrating both sides w.r.t. a, we get

$$I=-b.\frac{1}{b}\tan^{-1}\frac{a}{b}+c=-\tan^{-1}\frac{a}{b}+c.$$

When $a=0$, we do not get the value of (I) and hence we did not follow it.

However if we put $b=0$, then $I=0$ and $\tan^{-1}\infty=\pi/2$.

$$\therefore\quad 0=-\frac{\pi}{2}+c\ ;\quad \therefore\quad c=\frac{\pi}{2}.$$

$$\therefore\quad I=\frac{\pi}{2}-\tan^{-1}\frac{a}{b}=\cot^{-1}\frac{a}{b}=\tan^{-1}\frac{b}{a}.$$

$$\because\quad \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2} \text{ and } \cot^{-1}x=\tan^{-1}\frac{1}{x}.$$

Remember that $\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2}$ if b is +ive.

In case b is $-$ive say $-k$, then

$\sin bx=\sin(-kx)=-\sin kx$, where k is +ive.

$\therefore\quad \int_0^\infty \frac{\sin bx}{x}\,dx=-\int_0^\infty \frac{\sin kx}{x}\,dx$, where k is +ive$=-\left[\frac{\pi}{2}\right]$.

Hence $\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{-\pi}{2}$ **if b is $-$ive.**

Ex. 11. *Prove that* $\int_0^\infty \left(\frac{\sin ax}{x}\right)^2 dx=\frac{\pi a}{2}$, $a>0$.

$$I=\int_0^\infty \frac{1}{x^2}.\sin^2 ax\,dx.$$

Integrating by parts, we get

$$I=\left[-\frac{1}{x}\sin^2 ax\right]_0^\infty+a\int_0^\infty \frac{2\sin ax\cos ax}{x}\,dx$$

$$=0+a\int_0^\infty \frac{\sin 2ax}{x}\,dx=a\frac{\pi}{2}.$$

$\because$ we know from Ex. 10 that $\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2}$.

Ex. 12. *Evaluate* $\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx$ *where a and b are +ive.*

$$I=\frac{1}{2}\int_0^\infty \frac{\cos(b-a)x-\cos(b+a)x}{x^2}\,dx.$$

Integrating by parts,

$$I=-\frac{1}{x}\Big[\cos(b-a)x-\cos(b+a)x\Big]_0^\infty$$
$$+\int_0^\infty \frac{1}{x}\Big\{(b+a)\sin(b+a)x-(b-a)\sin(b-a)x\Big\}\,dx.$$

Clearly the first expression vanishes when $x=\infty$ and also when $x\to 0$, then

$$\text{Lt}\,\frac{1}{x}\,[\cos(b-a)x-\cos(b+a)x]=\frac{0}{0}$$
$$=\underset{x\to 0}{\text{Lt}}\,\frac{(b+a)\sin(b+a)x-(b-a)\sin(b-a)x}{1}=0.$$

$$\therefore\quad I=0+\int_0^\infty (b+a)\frac{\sin(b+a)x}{x}\,dx-\int_0^\infty (b-a)\frac{\sin(b-a)x}{x}\,dx$$

If **b > a**, then $(b-a)$ and $(b+a)$ are both +ive and we know from Ex. 10 that

$$\int_0^\infty \frac{\sin bx}{x}\,dx=\frac{\pi}{2}, \text{ if } b \text{ is +ive.}$$

$$\therefore\quad I=(b+a).\frac{\pi}{2}-(b-a)\frac{\pi}{2}=\pi a.$$

If **b < a**, then $(b-a)$ is $-$ive and we know that

$$\int_0^\infty \frac{\sin bx}{x}\,dx=-\frac{\pi}{2}, \text{ if } b \text{ is -ive.}$$

$$\therefore\quad I=(b+a)\frac{\pi}{2}-(b-a)\left(-\frac{\pi}{2}\right)=\pi b.$$

Ex. 13. $\int_0^\infty \frac{\cos ax}{x} \frac{\sin x}{} dx = \frac{\pi}{2}$,

if a is +ive and less than 1.

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(1+a)x + \sin(1-a)x}{x}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}\right] + \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{2}, \text{ by result Ex. 10.}$$

Ex. 14. *Prove that*

$$\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x} dx = \pi[\sqrt{(1+y)} - 1] \text{ when } y > 1.$$

(Bombay 59)

$I = \int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x} dx.$ Differentiate w.r.t. y.

$$\frac{dI}{dy} = \int_0^{\pi/2} \frac{1}{(1+y\sin^2 x).\sin^2 x} \sin^2 x\, dx = \int_0^{\pi/2} \frac{1}{1+y\sin^2 x} dx.$$

Divide above and below by $\cos^2 x$.

$$\therefore \quad \frac{dI}{dy} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{(1+\tan^2 x) + y\tan^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\, dx}{1+(y+1)\tan^2 x}.$$

Put $\sqrt{(1+y)}\tan x = t$; $\quad \therefore \sqrt{(1+y)}\sec^2 x\, dx = dt$ and adjust the limits.

$$\therefore \quad \frac{dI}{dy} = \frac{1}{\sqrt{(1+y)}}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{\sqrt{(1+y)}}\left[\tan^{-1} t\right]_0^\infty = \frac{\pi}{2\sqrt{(1+y)}}.$$

Integrating both sides w.r.t. y, the quantity w.r.t. which we differentiated original integral,

$I = \frac{\pi}{2}.2\sqrt{(1+y)} + c.$ When $y=0$, $I=0$; $\therefore c = -\pi$.

$$\therefore \quad I = \pi[\sqrt{(1+y)} - 1].$$

Ex. 15. *If $e^2 < 1$, prove that*

$$\int_0^{\pi/2} \log(1-e^2\sin^2\theta)\, d\theta = \pi \log\left[\frac{1+\sqrt{(1-e^2)}}{2}\right].$$

$I = \int_0^{\pi/2} \log(1-e^2\sin^2\theta)\, d\theta.$ Diff. w.r.t. e, we get

$$\frac{dI}{de}=\int_0^{\pi/2} \frac{1}{1-e^2 \sin^2 \theta}(-2e \sin^2 \theta)\, de$$

$$=\frac{2}{e}\int_0^{\pi/2} \frac{-e^2 \sin^2 \theta}{1-e^2 \sin^2 \theta}\, d\theta=\frac{2}{e}\int_0^{\pi/2} \frac{(1-e^2 \sin^2 \theta)-1}{1-e^2 \sin^2 \theta}\, d\theta$$

$$=\frac{2}{e}\int_0^{\pi/2}\left(1-\frac{1}{1-e^2 \sin^2 \theta}\right) d\theta$$

$$=\frac{2}{e}\frac{\pi}{2}-\frac{2}{e}\int_0^{\pi/2} \frac{\sec^2 \theta}{(1+\tan^2 \theta)-e^2 \tan^2 \theta}\, d\theta$$

$$=\frac{\pi}{e}-\frac{2}{e}\int_0^{\pi/2} \frac{\sec^2 \theta\, d\theta}{1+(1-e^2)\tan^2 \theta},\ e^2<1.$$

Put $\sqrt{(1-e^2)} \tan \theta=t$ and adjust the limits.

$$\frac{dI}{de}=\frac{\pi}{e}-\frac{2}{e}\cdot\frac{1}{\sqrt{(1-e^2)}}\int_0^{\infty} \frac{dt}{1+t^2}=\frac{\pi}{e}-\frac{2}{e\sqrt{(1-e^2)}}\left[\tan^{-1} t\right]_0^{\infty}$$

or

$$\frac{dI}{de}=\frac{\pi}{e}-\frac{2}{e\sqrt{(1-e^2)}}\cdot\frac{\pi}{2}.$$

Integrating both sides w.r.t. e,

$$I=\pi \log e-\pi\int \frac{de}{e\sqrt{(1-e^2)}}+C. \qquad \ldots(1)$$

But $\int \frac{de}{e\sqrt{(1-e^2)}}=\int \frac{\cos z\, dz}{\sin z \cos z}$ where $e=\sin z$

$$=\int \operatorname{cosec} z\, dz=\log\ (\operatorname{cosec} z+\cot z)$$

$$=-\log\left[\frac{1}{\sin z}+\frac{\sqrt{(1-\sin^2 z)}}{\sin z}\right]=-\log\left[\frac{1+\sqrt{(1-e^2)}}{e}\right]. \qquad \ldots(2)$$

Hence from (1) by the help of (2), we get

$$I=\pi \log e+\pi \log \frac{1+\sqrt{(1-e^2)}}{e}+C=\pi \log \frac{1+\sqrt{(1-e^2)}}{e}\cdot e+C.$$

$$\because\ \log m+\log n=\log mn$$

or

$$I=\pi \log \{1+\sqrt{(1-e^2)}\}+C.$$

When $e=0$, $I=0$; $\therefore\ 0=\pi \log 2+C$ or $C=-\pi \log 2$.

$$\therefore\ I=\pi \log [1+\sqrt{(1-e^2)}]-\pi \log 2$$

$$=\pi \log\left[\frac{1+\sqrt{(1-e^2)}}{2}\right]$$

Proved.

Ex. 16. Prove that

$$\int_0^\infty \frac{\tan^{-1}\alpha x \tan^{-1}\beta x}{x^2}\,dx=\frac{1}{2}\pi\log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha.\beta^\beta}\right].$$

(Delhi Hons. 56 ; Agra 59 ; Gujrat 59)

$$\frac{dI}{d\alpha}=\int_0^\infty \frac{x}{(1+\alpha^2x^2)}\cdot\frac{\tan^{-1}\beta x}{x^2}\,dx=\int_0^\infty \frac{\tan^{-1}\beta x}{x(1+\alpha^2x^2)}dx. \quad \ldots(1)$$

Differentiating again w r t. β, we get

$$\frac{d^2I}{d\beta.d\alpha}=\int_0^\infty \frac{1}{x(1+\alpha^2x^2)}\cdot\frac{x}{(1+\beta^2x^2)}\,dx=\int_0^\infty \frac{dx}{(1+\alpha^2x^2)(1+\beta^2x^2)}$$

Split into partial fractions by method of suppression.

$$\frac{d^2I}{d\beta\, d\alpha}=\frac{1}{(\alpha^2-\beta^2)}\int_0^\infty\left(\frac{\alpha^2}{1+\alpha^2x^2}-\frac{\beta^2}{1+\beta^2x^2}\right)dx$$

$$=\frac{1}{\alpha^2-\beta^2}\Big[\alpha\tan^{-1}\alpha x-\beta\tan^{-1}\beta x\Big]_0^\infty$$

$$=\frac{1}{\alpha^2-\beta^2}\left(\alpha.\frac{\pi}{2}-\beta.\frac{\pi}{2}\right)$$

or

$$\frac{d^2I}{d\beta\, d\alpha}=\frac{\pi}{2}\cdot\frac{1}{\alpha+\beta}.$$

Integrating w.r.t. β, we get

$$\frac{dI}{d\alpha}=\frac{\pi}{2}\log(\alpha+\beta)+c$$

where c is a constant of integration independent of β.

When $\beta=0$, $\dfrac{dI}{d\alpha}=0$; $\therefore$ $c=-\dfrac{\pi}{2}\log\alpha$ from (1).

$$\therefore\quad \frac{dI}{d\alpha}=\frac{\pi}{2}[\log(\alpha+\beta)-\log\alpha]$$

Integrating again w.r.t. α, we get

$$I=\frac{\pi}{2}\left[\int\log(\alpha+\beta)\,d\alpha-\int\log\alpha\,d\alpha\right]$$

$$=\frac{\pi}{2}\left[\alpha.\log(\alpha+\beta)-\int\frac{\alpha}{\alpha+\beta}\,d\alpha-\left\{\alpha.\log\alpha-\int\alpha.\frac{1}{\alpha}d\alpha\right\}\right]+D.$$

$$=\frac{\pi}{2}\left[\alpha \log(\alpha+\beta)-\{\alpha-\beta \log(\alpha+\beta)\}-\alpha \log \alpha+\alpha\right]+D$$

$$=\frac{\pi}{2}\left[(\alpha+\beta)\log(\alpha+\beta)-\alpha \log \alpha\right]+D.$$

When $\alpha=0$, $I=0$, $\therefore \; 0=\frac{\pi}{2}(\beta \log \beta)+D.$

$$\therefore \quad D=-\frac{\pi}{2}\beta \log \beta \; ;$$

$$\therefore \quad I=\frac{\pi}{2}\left[(\alpha+\beta)\log(\alpha+\beta)-\alpha \log \alpha-\beta \log \beta\right]$$

$$=\frac{\pi}{2}\left[\log(\alpha+\beta)^{\alpha+\beta}-\log \alpha^{\alpha}.\beta^{\beta}\right]$$

$$=\frac{\pi}{2}\log \frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^{\alpha}.\beta^{\beta}}.$$

§ 3. Integration under the sign of integration. We shall illustrate this method by taking some suitable examples.

Ex. 1. *We know that* $\int_0^1 x^{a-1}\,dx=\left[\frac{x^a}{a}\right]_0^1=\frac{1}{a}.$

Integrating both sides w.r.t. a within limits α to β, we get

$$\int_0^1\int_\alpha^\beta x^{a-1}\,da\,dx=\int_\alpha^\beta \frac{da}{a}.$$

Remember that $\int a^x\,dx=\frac{a^x}{\log a}.$

$$\therefore \quad \int_0^1\left[\frac{x^{a-1}}{\log x}\right]_\alpha^\beta dx=\Big[\log a\Big]_\alpha^\beta$$

or
$$\int_0^1 \frac{x^{\beta-1}-x^{\alpha-1}}{\log x}\,dx=\log \beta-\log \alpha=\log \frac{\beta}{\alpha}. \qquad \ldots(1)$$

Another form.

We know that

$$\int_0^\infty e^{-ax}\,dx=-\frac{1}{a}\left[e^{-ax}\right]_0^\infty=-\frac{1}{a}(0-1)=\frac{1}{a}.$$

Integrating both sides w.r.t. a within limits α to β,

$$\int_0^\infty \int_\alpha^\beta e^{-ax}\, da\, dx = \int_\alpha^\beta \frac{da}{a}$$

$$= \int_0^\infty \left[\frac{e^{-ax}}{-x}\right]_\alpha^\beta dx = \Big[\log a\Big]_\alpha^\beta$$

or $$\int_0^\infty \frac{e^{-\beta x} - e^{-\alpha x}}{-x}\, dx = \log \beta - \log \alpha = \log \frac{\beta}{\alpha}.$$

or $$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\, dx = \log \frac{\beta}{\alpha}. \qquad \ldots(2)$$

Result (2) could also be deduced from result (1) by making the substitution $x = e^{-z}$; $\therefore\ dx = -e^{-z}\, dz = -x\, dz$.

Also $\qquad -z = \log x$.

Also when $x = 1$, $z = 0$ and when $x = 0$, $z = \infty$.

$$\int_0^1 \frac{x^\beta\ x^{-1} - x^\alpha\ x^{-1}}{\log x}\, dx = \log \frac{\beta}{\alpha}$$

or $$\int_\infty^0 \frac{e^{-\beta z} - e^{-\alpha z}}{x.(-z)}(-x\, dz) = \log \frac{\beta}{\alpha}$$

$$= \int_0^\infty \frac{e^{-\alpha z} - e^{-\beta z}}{z} dz = \log \frac{\beta}{\alpha},$$

which is of the same form as (2).

Note. If we start with $\int_0^1 x^a\, dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$ and proceeding as above, we can prove that

$$\int_0^1 \frac{x^\beta - x^\alpha}{\log x}\, dx = \log \frac{\beta+1}{\alpha+1}. \qquad (Gujarat\ 52)$$

Again if we put $\alpha = 0$, then $\int_0^1 \frac{x^\beta - 1}{\log x}\, dx = \log(\beta+1)$ ..(3)

Alternative method by differentiation for the last result.

Let $$I = \int_0^1 \frac{x^\beta - 1}{\log x}\, dx.$$

$$\therefore \quad \frac{dI}{d\beta} = \int_0^1 \frac{x^\beta \log x}{\log x}\, dx = \left[\frac{x^{\beta+1}}{\beta+1}\right]_0^1 = \frac{1}{\beta+1}.$$

Integrating w.r.t. β, we get

$$I=\log(\beta+1)+c.$$

When $\beta=0$, $I=0$; $\therefore$ $c=0$.

Hence $I=\log(\beta+1)$, which is same as (3).

Ex. 2. *Evaluate*

$$\int_0^\infty e^{-ax}\sin mx\,dx \text{ and } \int_0^\infty e^{-ax}\cos mx\,dx$$

and by applying the principle of integration under the sign of integration deduce other results.

We know that $\displaystyle\int_0^\infty e^{-ax}\sin mx\,dx=\frac{m}{a^2+m^2}$. ...(1)

[Refer Ex. 3 P. 29]

Integrating both sides w r t. a, within limits α to β,

$$\int_0^\infty\int_\alpha^\beta (e^{-ax}.da)\sin mx\,dx=\int_\alpha^\beta \frac{m}{a^2+m^2}\,da$$

or $$\int_0^\infty\left[\frac{e^{-ax}}{-x}\right]_\alpha^\beta \sin mx\,dx=\left[\tan^{-1}\frac{a}{m}\right]_\alpha^\beta$$

or $$\int_0^\infty\left[\frac{e^{-\alpha x}-e^{-\beta x}}{x}\right]\sin mx\,dx=\tan^{-1}\frac{\beta}{m}-\tan^{-1}\frac{\alpha}{m}. \quad ...(2)$$

If we put $\alpha=0$, *i.e.*, $e^{-\alpha x}=1$, $\beta=\infty$, *i.e.*, $e^{-\beta x}=0$, then we get

$$\int_0^\infty \frac{\sin mx}{x}\,dx=\tan^{-1}\infty=\frac{\pi}{2}. \quad ...(3)$$

The above result we have already proved in Ex. 10, P. 37.

Note. See also Ex. 7 P. 50 for independent proof of (3).

Again integrating both sides of (1) w.r t. m, we get

$$\int_0^\infty\int_\alpha^\beta (\sin mx\,dm)\,e^{-ax}.dx=\int_\alpha^\beta \frac{m}{a^2+m^2}\,dm$$

or $$\int_0^\infty\left[-\frac{\cos mx}{x}\right]_\alpha^\beta e^{-ax}\,dx=\frac{1}{2}\left[\log(a^2+m^2)\right]_\alpha^\beta$$

or $$\int_0^\infty \frac{\cos\alpha x-\cos\beta x}{x}\,e^{-ax}\,dx=\frac{1}{2}\log\frac{\beta^2+a^2}{\alpha^2+a^2}. \quad ...(4)$$

If we put $a=0$ in the above, then

$$\int_0^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx = \frac{1}{2} \log \frac{\beta^2}{\alpha^2} = \log \frac{\beta}{\alpha}. \quad \ldots(5)$$

(Vikram 62)

Again $$\int_0^\infty e^{-ax} \cos mx\, dx = \frac{a}{a^2+m^2}. \quad \ldots(6)$$

[Ex. 3 (c) P. 30]

Integrating both sides w.r.t. m, within limits α to β,

$$\int_0^\infty \int_\alpha^\beta (\cos mx\, dm)\, e^{-ax}\, dx = \int_\alpha^\beta \frac{a}{a^2+m^2}\, dm,$$

$$\int_0^\infty \left[\frac{\sin mx}{x}\right]_\alpha^\beta e^{-ax}\, dx = \left[\tan^{-1} \frac{m}{a}\right]_\alpha^\beta$$

or $$\int_0^\infty \frac{\sin \beta x - \sin \alpha x}{x}\, e^{-ax}\, dx = \tan^{-1} \frac{\beta}{a} - \tan^{-1} \frac{\alpha}{a}. \quad \ldots(7)$$

Again integrating both sides of (6) w.r.t. (a), we get

$$\int_0^\infty \int_\alpha^\beta (e^{-ax}\, da) \cos mx\, dx = \int_\alpha^\beta \frac{a}{a^2+m^2}\, da$$

$$= \int_0^\infty \left[\frac{e^{-ax}}{-x}\right]_\alpha^\beta \cos mx\, dx = \frac{1}{2} \left[\log (a^2+m^2)\right]_\alpha^\beta$$

or $$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos mx\, dx = \frac{1}{2} \log \frac{\beta^2+m^2}{\alpha^2+m^2}.$$

Some Important Questions

Ex. 3. Prove that $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$. **[Remember]**

(Nagpur 63, 62 ; Karnatak 62 ; Vikram 64 ; Sagar 64 ; Poona 50, 55 ; Agra 52, 55, 58)

Let $$I = \int_0^\infty e^{-x^2}\, dx.$$

Put $x = \alpha y$; $\therefore\ dx = \alpha\, dy$.

$$\therefore\ I = \int_0^\infty e^{-\alpha^2 y^2} \alpha\, dy.$$

Now multiplying both sides by $e^{-\alpha^2}$, we get

$$I . e^{-\alpha^2} = \int_0^\infty e^{-\alpha^2 (1+y^2)} \alpha\, dy.$$

Integrating both sides w. r. t. α (Note) within the limits 0 to infinity,

$$I\int_0^{\infty} e^{-\alpha^2}\,d\alpha=\int_0^{\infty}\int_0^{\infty} e^{-\alpha^2(1+y^2)}\,\alpha.\,d\alpha\,dy$$

$$=\int_0^{\infty}\left[-\frac{1}{2}\frac{e^{-\alpha^2(1+y^2)}}{(1+y^2)}\right]_0^{\infty} dy$$

$$=\int_0^{\infty}\frac{1}{2}\frac{dy}{1+y^2}=\frac{1}{2}\left[\tan^{-1} y\right]_0^{\infty}=\frac{1}{2}\,\frac{\pi}{2}.$$

But $\int_0^{\infty} e^{-\alpha^2}\,d\alpha=I$; $\therefore$ L.H.S.$=I.I=I^2$.

Hence $I^2=\frac{\pi}{4}$; $\therefore$ $I=\int_0^{\infty} e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$.

Deduction. $\int_0^{\infty} e^{-k^2x^2}\,dx=\frac{1}{2k}\sqrt{\pi}$.

Put $kx=z$.

$$\therefore\quad I=\int_0^{\infty} e^{-z^2}.\frac{dz}{k}=\frac{1}{k}.\frac{1}{2}.\sqrt{\pi}=\frac{1}{2k}\sqrt{\pi}.$$

Ex. 4. **Prove that**

$$\int_0^{\infty} e^{-(x^2+a^2/x^2)b^2}\,dx=\frac{\sqrt{\pi}}{2b}\,e^{-2ab^2}.$$

Hence deduce that

$$\int_0^{\infty} e^{-(x^2+a^2/x^2)}\,dx=\frac{\sqrt{\pi}}{2}\,e^{-2a}.$$

(Rajputana 53, 55, 58 ; Agra 60)

Let $I=\int_0^{\infty} e^{-(x^2+a^2/x^2)\,b^2}\,dx.$

$$\therefore\quad \frac{dI}{da}=\int_0^{\infty} e^{-(x^2+a^2/x^2)\,b^2}\left(-\frac{2ab^2}{x^2}\right)dx.$$

Now put $\frac{a}{x}=z$; $\therefore$ $-\frac{a}{x^2}\,dx=dz$ and adjust the limits.

$$\therefore \quad \frac{dI}{da}=2b^2\int_{\infty}^{0} e^{-(a^2/z^2+z^2)\,b^2}\,dz$$

$$=-2b^2\int_0^{\infty} e^{-(a^2/z^2+z^2)\,b^2}\,dz$$

or $$\frac{dI}{da}=-2b^2 I,$$ variables separable.

$$\therefore \quad \int \frac{dI}{I}=-2b^2\int da.$$

$$\therefore \quad \log I=-2b^2 a+\log c,$$

where $\log c$ is constant of integration.

$$\therefore \quad \log\frac{I}{c}=-2b^2 a \quad \text{or} \quad I=ce^{-2b^2 a}.$$

In order to find c we use that when $a=0$, then

$$I=\int_0^{\infty} e^{-x^2b^2}\,dx=\frac{\sqrt{\pi}}{2b}$$ by deduction Ex. 3 P. 47.

$$\therefore \quad \frac{\sqrt{\pi}}{2b}=c.1.$$

Putting for c, we get

$$I=\int_0^{\infty} e^{-(x^2+a^2/x^2)\,b^2}\,dx=\frac{\sqrt{\pi}}{2b}\,e^{-2b^2 a}.$$

Deduction. Putting $b=1$, we get

$$\int_0^{\infty} e^{-(x^2+a^2/x^2)}\,dx=$$

Ex. 5. Prove that

$$\int_0^{\infty} e^{-a^2x^2}\cos 2bx\,dx$$

ar 63 ; Raj

Let $-a^2x^2$ co

$-a^2$

Integrating by parts, we get

$$\frac{dI}{db}=\left[\frac{e^{-a^2x^2}}{a^2}\ \sin 2bx\right]_0^\infty-\int_0^\infty \frac{e^{-a^2x^2}}{a^2}.2b\cos 2bx\ dx$$

$$\left(\text{Integral of } -2xe^{-a^2x^2} \text{ is } \frac{e^{-a^2x^2}}{a^2}\right)$$

or $$\frac{dI}{db}=0-\frac{2b}{a^2}I, \text{ variables separable.}$$

$$\therefore\quad \int\frac{dI}{I}=\int-\frac{2b}{a^2}\,db \text{ or } \log I=-\frac{b^2}{a^2}+\log c.$$

$$\therefore\quad \log\frac{I}{c}=-\frac{b^2}{a^2} \text{ or } I=ce^{-b^2/a^2}$$

When $b=0$, then

$$I=\int_0^\infty e^{-a^2x^2}.1.dx=\frac{\sqrt{\pi}}{2a} \text{ by deduction Ex. 3.}$$

$$\therefore\quad I=\int_0^\infty e^{-a^2x^2}\cos 2bx\ dx=\frac{\sqrt{\pi}}{2a}e^{-b^2/a^2}.$$

Ex. 6. Prove that

$$\int_0^\infty e^{-ax^2}\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}}\ (a>0)$$

and deduce that

$$\int_0^\infty x^{2n}e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2^{n+1}}\,[1.3.5\ldots(2n-1)].$$

(Sagar 64)

We have proved in deduction Ex. 3 P. 47 that

$$\int_0^\infty e^{-k^2x^2}=\frac{\sqrt{\pi}}{2k}.$$

Putting $k=\sqrt{a}$ i.e. $k^2=a$, we prove the first part.

Now differentiate the equation

$$\int_0^\infty e^{-ax^2}\,dx=\frac{\sqrt{\pi}}{2\sqrt{a}} \qquad \ldots(1)$$

n times w.r.t. a and remember that $\frac{d^n}{dx^n}(e^{mx})=m^ne^{mx}$

and $$\frac{d^n}{dx^n}\left(\frac{1}{\sqrt{x}}\right)=(-1)^n.\tfrac{1}{2}.\tfrac{3}{2}.\tfrac{5}{2}\ldots\frac{2n-1}{2}\frac{1}{x^{n+1/2}}.$$

and hence deduce the following integrals :—

(a) $\int_0^\infty \frac{\sin mx}{x(a^2+x^2)} dx = \frac{\pi}{2a^2}(1-e^{-ma})$.

(b) $\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$.

1st Method.

Let $$I = \int_0^\infty \frac{\sin mx}{x(a^2+x^2)} dx. \quad \ldots(1)$$

$$\therefore \quad \frac{dI}{dm} = \int_0^\infty \frac{\cos mx}{a^2+x^2} dx \quad \ldots(2)$$

and $$\frac{d^2I}{dm^2} = -\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx \quad \ldots(3)$$

$$= -\int_0^\infty \frac{(x^2+a^2-a^2)}{x(a^2+x^2)} \sin mx \, dx \qquad \textit{(Note)}$$

$$= -\int_0^\infty \left[\frac{\sin mx}{x} - \frac{a^2 \sin mx}{x(a^2+x^2)}\right] dx$$

or $$\frac{d^2I}{dm^2} = -\frac{\pi}{2} + a^2 I \text{ by (1) and result Ex. 7, P. 50}$$

or $$\frac{d^2I}{dm^2} = a^2\left(I - \frac{\pi}{2a^2}\right).$$

Put $$I - \frac{\pi}{2a^2} = v; \quad \therefore \quad \frac{d^2I}{dm^2} = \frac{d^2v}{dm^2}.$$

Hence we have $\frac{d^2v}{dm^2} - a^2v = 0$ or $(D^2-a^2)v = 0$,

where $D = \frac{d}{dm}$.

Solution of above differential equation is

$$v = Ae^{am} + Be^{-am} \quad \text{or} \quad I - \frac{\pi}{2a^2} = Ae^{am} + Be^{-am}$$

or $$I = \frac{\pi}{2a^2} + Ae^{am} + Be^{-am}. \quad \ldots(4)$$

Also $$\frac{dI}{dm}=Aae^{am}-Bae^{-am}. \qquad \ldots($$

Again when $m=0$, $I=0$.

$$\frac{dI}{dm}=\int_0^\infty \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^\infty=\frac{\pi}{2a} \text{ by (1) and (2}$$

Putting the above results in (4) and (5), we get

$$0=\frac{\pi}{2a^2}+A+B, \text{ and } \frac{\pi}{2a}=a\,(A-B).$$

$$\therefore \quad A+B=-\frac{\pi}{2a^2} \text{ and } A-B=\frac{\pi}{2a^2}.$$

$$\therefore \quad A=0 \text{ and } B=-\frac{\pi}{2a^2}.$$

$$\therefore \quad I=\frac{\pi}{2a^2}-\frac{\pi}{2a^2}e^{-am} \text{ by (4)}$$

or $$\int_0^\infty \frac{\sin mx}{x\,(a^2+x^2)}=\frac{\pi}{2a^2}\,(1-e^{-ma}) \text{ by (1)} \qquad \ldots(6)$$

and $$\frac{dI}{dm}=\frac{\pi}{2a^2}\,ae^{-am}$$

or $$\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx=\frac{\pi}{2a}\,e^{-ma} \text{ by (2).} \qquad \ldots(7)$$

Again $$\frac{d^2I}{dm^2}=\frac{\pi}{2a}\,(-a)\,e^{-am}=-\frac{\pi}{2}\,e^{-ma}$$

or $$\int_0^\infty -\frac{x\sin mx\,dx}{a^2+x^2}=-\frac{\pi}{2}\,e^{-ma} \text{ by (3)}$$

or $$\int_0^\infty \frac{x\sin mx\,dx}{a^2+x^2}=\frac{\pi}{2}\,e^{-ma}. \qquad \ldots(8)$$

Putting $a=1$ in (6), (7) and (8), we get

$$\int_0^\infty \frac{\sin mx}{x\,(1+x^2)}\,dx=\frac{\pi}{2}\,(1-e^{-m}),\ \int_0^\infty \frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m},$$

$$\int \frac{x\sin mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m}.$$

Alternative method.

We know that

$$\int_0^\infty 2ze^{-(a^2+x^2)\,z^2}\,dz = -\frac{1}{a^2+x^2}\left[e^{-(a^2+x^2)\,z^2}\right]_0^\infty = \frac{1}{a^2+x^2}.$$

Multiplying both sides by $\cos mx$; we get

$$\frac{\cos mx}{a^2+x^2} = \int_0^\infty \cos mx\,.\,2ze^{-(a^2+x^2)\,z^2}\,dz.$$

Integrate both sides w.r.t. x within limits 0 to ∞ and let

$$\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx = I.$$

$$\therefore\quad I = \int_0^\infty \int_0^\infty \cos mx\,.\,2ze^{-(a^2+x^2)\,z^2}.dz.dx.$$

$$= \int_0^\infty 2ze^{-a^2z^2} \int_0^\infty (e^{-x^2z^2}\cos mx\,dx)\,dz.$$

But we know from Ex. 5 P. 48 that

$$\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2a}\,e^{-b^2/a^2}.$$

$$\therefore\quad I = \int_0^\infty 2ze^{-a^2z^2}\,\frac{\sqrt{\pi}}{2z}\,e^{-m^2/4z^2}\,dz,\quad \because\quad a=z \text{ and } b=\frac{m}{2}.$$

or $$I = \sqrt{\pi}\int_0^\infty e^{-(a^2z^2+m^2/4z^2)}\,dz$$

$$= \sqrt{\pi}\int_0^\infty e^{-\left(z^2+\frac{m^2}{4a^2z^2}\right)a^2}\,dz.$$

But we know that Ex. 4 P. 47, that

$$\int_0^\infty \left\{e^{-(x^2+a^2/x^2)\,b^2}\right\} = \frac{\sqrt{\pi}}{2b}\,e^{-2b^2a}.$$

$$\therefore\quad I = \sqrt{\pi}\,.\,\frac{\sqrt{\pi}}{2a}\,e^{-2a^2.m/2a} = \frac{\pi}{2a}\,e^{-am}$$

or $$\int_0^\infty \frac{\cos mx}{a^2+x^2} = \frac{\pi}{2a}\,e^{-ma}, \qquad \ldots(1)$$

$\therefore \quad I=\cos br.2\int_0^\infty \frac{\cos yr}{y^2+a^2}dy=2\cos br\,\frac{\pi}{2a}\,e^{-ar}$

$=\frac{\pi}{a}.e^{-ar}\cos br$ by the given result.

(b) Prove as in (a).

(c) $\int_0^\infty \frac{\cos mx}{(a^2+x^2)}dx=\frac{\pi}{2a}\,e^{-ma}$. (given)

Differentiating both sides w.r t. a, we get

$$\int_0^\infty -\frac{\cos mx}{(a^2+x^2)^2}(2a)\,dx=\frac{\pi}{2}\left[-\frac{1}{a^2}\,e^{-ma}+\frac{1}{a}\,(-m)e^{-ma}\right].$$

$$\therefore \quad \int_0^\infty \frac{\cos mx}{(a^2+x^2)^2}\,dx=\frac{\pi}{4}\left[\frac{m}{a^2}+\frac{1}{a^3}\right]e^{-ma}.$$

(d) Putting $a=1$ in the given result, we have

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m}$$

Put $x=\tan\theta$. $\therefore$ $dx=\sec^2\theta\,d\theta$, and adjust the limits.

$$\therefore \quad \int_0^{\pi/2}\frac{\cos(m\tan\theta)}{(1+\tan^2\theta)}\sec^2\theta\,d\theta=\frac{\pi}{2}\,e^{-m}$$

or $\quad \int_0^{\pi/2}\cos(m\tan\theta)\,d\theta=\frac{\pi}{2}\,e^{-m}.$

Ex. 11. *Evaluate* $\int_0^\infty \cos x \log\frac{(x^2+\beta^2)}{(x^2+\alpha^2)}\,dx.$

$$I=\int_0^\infty \cos x\,[\log(x^2+\beta^2)-\log(x^2+\alpha^2)]\,dx.$$

$$\therefore \quad \frac{dI}{d\alpha}=\int_0^\infty \cos x.\frac{-2\alpha}{x^2+\alpha^2}\,dx=-2\alpha\int_0^\infty \frac{\cos x}{x^2+\alpha^2}\,dx$$

$=-2\alpha\left[\frac{\pi}{2\alpha}\,e^{-\alpha}\right]=-\pi e^{-\alpha}.$ by Ex. 8 P. 50

Integrating both sides w.r.t. α, we get

$$I=\pi e^{-\alpha}+c,$$

where c is a constant which is independent of α.

When $\alpha=\beta$, $I=0$; $\therefore$ $c=-\pi e^{-\beta}$; $\therefore$ $I=\pi(e^{-\alpha}-e^{-\beta}$

Dfferentiating (1) w.r.t. m, we get

$$\int_0^\infty -\frac{x \sin mx}{x\,(a^2+x^2)}=\frac{\pi}{2a}\,(-a)\,e^{-ma}.$$

$$\therefore \quad \int_0^\infty \frac{x \sin mx}{a^2+x^2}\,dx=\frac{\pi}{2}\,e^{-ma}.$$

Again integrating (1) w.r.t. m within limits 0 to m, we get

$$\int_0^\infty \left[\frac{\sin mx}{x\,(a^2+x^2)}\right]_0^m dx=\frac{\pi}{2a}\left[\frac{e^{-ma}}{-a}\right]_0^m$$

or

$$\int_0^\infty \frac{\sin mx}{x\,(a^2+x^2)}\,dx=\frac{\pi}{2a^2}\,(1-e^{-ma}).$$

Ex. 9. *Having given that* $\int_0^\infty \frac{\cos mx}{a^2+x^2}\,dx=\frac{\pi}{2a}\,e^{-ma}$, *prove the following results :—*

(a) $\displaystyle\int_{-\infty}^\infty \frac{\cos rx}{(x-b)^2+a^2}\,dx=\frac{\pi}{a}\,e^{-ar}\cos br.$

(b) $\displaystyle\int_{-\infty}^\infty \frac{\sin rx}{(x-b)^2+a^2}\,dx=\frac{\pi}{a}\,e^{-ar}\sin br,$

(c) $\displaystyle\int_0^\infty \frac{\cos mx}{(a^2+x^2)^2}\,dx=\frac{\pi}{4}\left[\frac{m}{a^2}+\frac{1}{a^3}\right]e^{-ma},$

(d) $\displaystyle\int_0^{\pi/2} \cos\,(m \tan\theta)\,d\theta=\frac{\pi}{2}\,e^{-m}.$

(a) Put $x-b=y$, $\therefore$ $x=b+y$ and $dx=dy$.

$$\therefore \quad I=\int_{-\infty}^\infty \frac{\cos\,(br+yr)}{y^2+a^2}\,dy$$

$$=\cos br\int_{-\infty}^\infty \frac{\cos yr}{y^2+a^2}\,dy-\sin br\int_{-\infty}^\infty \frac{\sin yr}{y^2+a^2}\,dy.$$

Now $\int_{-a}^a f(x)\,dx=0$, if $f(x)$ is an odd function of x

$$=2\int_0^a f(x)\,dx$$

if $f(x)$ is an even function of x.

$\therefore \quad I=\cos br . 2\int_0^\infty \frac{\cos vr}{y^2+a^2}dy=2\cos br\,\frac{\pi}{2a}\,e^{-ar}$

$=\frac{\pi}{a}.e^{-ar}\cos br$ by the given result.

(b) Prove as in (a).

(c) $\int_0^\infty \frac{\cos mx}{(a^2+x^2)}dx=\frac{\pi}{2a}\,e^{-ma}$. (given)

Differentiating both sides w.r.t. a, we get

$\int_0^\infty -\frac{\cos mx}{(a^2+x^2)^2}(2a)\,dx=\frac{\pi}{2}\left[-\frac{1}{a^2}\,e^{-ma}+\frac{1}{a}\,(-m)e^{-ma}\right].$

$\therefore \quad \int_0^\infty \frac{\cos mx}{(a^2+x^2)^2}\,dx=\frac{\pi}{4}\left[\frac{m}{a^2}+\frac{1}{a^3}\right]e^{-ma}.$

(d) Putting $a=1$ in the given result, we have

$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx=\frac{\pi}{2}\,e^{-m}$

Put $x=\tan\theta$. $\therefore \quad dx=\sec^2\theta\,d\theta$, and adjust the limits.

$\therefore \quad \int_0^{\pi/2}\frac{\cos(m\tan\theta)}{(1+\tan^2\theta)}\sec^2\theta\,d\theta=\frac{\pi}{2}\,e^{-m}$

or $\quad \int_0^{\pi/2}\cos(m\tan\theta)\,d\theta=\frac{\pi}{2}\,e^{-m}.$

Ex. 11. *Evaluate* $\int_0^\infty \cos x\log\frac{(x^2+\beta^2)}{(x^2+\alpha^2)}\,dx.$

$I=\int_0^\infty \cos x\,[\log(x^2+\beta^2)-\log(x^2+\alpha^2)]\,dx.$

$\therefore \quad \frac{dI}{d\alpha}=\int_0^\infty \cos x.\frac{-2\alpha}{x^2+\alpha^2}\,dx=-2\alpha\int_0^\infty\frac{\cos x}{x^2+\alpha^2}\,dx$

$=-2\alpha\left[\frac{\pi}{2\alpha}\,e^{-\alpha}\right]=-\pi e^{-\alpha}.$ by Ex. 8 P. 50

Integrating both sides w.r.t. α, we get

$I=\pi e^{-\alpha}+c,$

where c is a constant which is independent of α.

When $\alpha=\beta$, $I=0$; $\therefore \; c=-\pi e^{-\beta}$; $\because \; I=\pi\,(e^{-\alpha}-e^{-\beta}$

§ 4. $\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx$

In the above integral $\frac{f(x)}{F(x)}$ *is rational algebraic function in which the numerator* $f(x)$ *is at least two degrees less than that of the denominator* $F(x)$. *Also all the roots of the denominator* $F(x)=0$ *are supposed to be imaginary.*

We are given that $F(x)=0$ has all its roots imaginary i.e. $F(x)$ is of such a form that it will never vanish for any real value of x. Therefore the function $\frac{f(x)}{F(x)}$ is always finite for all real values of x.

$$\therefore \quad \int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx = \underset{\epsilon \to 0}{\text{Lt}} \int_{-1/\mu\epsilon}^{1/\nu\epsilon} \frac{f(x)}{F(x)}\, dx.$$

Now we know that imaginary roots occur in conjugate pairs and let one of the pair of roots of $F(x)=0$ be $a \pm ib$ so that the corresponding factors of $F(x)$ are $x-(a+ib)$ and $x-(a-ib)$ i.e. $(x-a)-ib$ and $(x-a)+ib$. Therefore the partial fractions corresponding to these two factors of the denominator are of the form

$$\frac{L-iM}{(x-a)-ib} \text{ and } \frac{L+iM}{(x-a)+ib}.$$

Now $$\frac{L-iM}{(x-a)-ib}+\frac{L+iM}{(x-a)+ib}=\frac{2L(x-a)+2Mb}{(x-a)^2+b^2}. \quad \ldots(1)$$

Again $$\int_{-\infty}^{\infty} \frac{2Mb}{(x-a)^2+b^2}\, dx = \underset{\epsilon \to 0}{\text{Lt}} \int_{-1/\mu\epsilon}^{1/\nu\epsilon} \frac{2Mb}{(x-a)^2+b^2}\, dx$$

$$=2M\left[\tan^{-1}\frac{x-a}{b}\right]_{-1/\mu\epsilon}^{1/\nu\epsilon}=2M\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=2M\pi \quad \ldots(2)$$

and $$\int_{-\infty}^{\infty} \frac{2L(x-a)}{(x-a)^2+b^2} = \underset{\epsilon \to 0}{\text{Lt}} \int_{-1/\mu\epsilon}^{1/\nu\epsilon} \frac{2L(x-a)\, dx}{(x-a)^2+b^2}$$

$$= \underset{\epsilon \to 0}{\text{Lt}}\; L\left[\log\left\{(x-a)^2+b^2\right\}\right]_{-1/\mu\epsilon}^{1/\nu\epsilon}.$$

$$=\underset{\epsilon\to 0}{\text{Lt}}\ L\log\left[\frac{\mu^2}{\nu^2}\frac{(1-a\nu\epsilon)^2+b^2\nu^2\epsilon^2}{(1+a\mu\epsilon)^2+b^2\mu^2\epsilon^2}\right]$$

$$=L\log\frac{\mu^2}{\nu^2}=2L\log\frac{\mu}{\nu}\text{ as }\epsilon\to 0. \quad \ldots(3)$$

$$\therefore\ \int_{-\infty}^{\infty}\frac{2L(x-a)+2Mb}{(x-a)^2+b^2}\,dx=2L\log\frac{\mu}{\nu}+2\pi M$$

from (1), (2), (3).

Above gives us the integral corresponding to one pair of conjugate roots of $F(x)=0$. If $F(x)=0$ be of $2n$ degree, then we shall have n conjugate pairs and the corresponding values of L and M in the partial fractions be denoted by $L_1, L_2, L_3 \ldots L_n$ and $M_1, M_2, M_3 \ldots M_n$; then as shown above for the integral corresponding to one pair, we have

$$\int_{-\infty}^{\infty}\frac{f(x)}{F(x)}\,dx=\left(2L_1\log\frac{\mu}{\nu}+2\pi M_1\right)+\left(2L_2\log\frac{\mu}{\nu}+2\pi M_2\right)+\ldots+\left(2L_n\log\frac{\mu}{\nu}+2\pi M_n\right)$$

$$=2(L_1+L_2+\ldots L_n)\log\frac{\mu}{\nu}+2\pi(M_1+M_2+\ldots+M_n) \quad \ldots(4)$$

Also by (1),

$$\frac{f(x)}{F(x)}=\frac{2L_1(x-a_1)+2M_1b_1}{(x-a_1)^2+b^2}+\ldots+\frac{2L_n(x-a_n)+2M_nb_n}{(x-a_n)^2+b_n^2}.$$

Multiplying both sides by $F(x)$, we get

$$f(x)=[2L_1(x-a_1)+2M_1b_1]\text{ (a function of degree } x^{2n-2})+\ldots \quad \ldots(5)$$

But $f(x)$ is of at least two degrees less than that of $F(x)$ whose degree is $2n$. Hence $f(x)$ will not have the term of x^{2n-1}, whereas the coefficient of x^{2n-1} in R.H.S. will be $2L_1+2L_2+\ldots+2L_n$.

Hence equating the coefficients of x^{2n-1} in (5), we get

$$0=2(L_1+L_2+\ldots+L_n) \quad \ldots(6)$$

Hence from (4) by the help of (6), we get

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx = 2\pi\,(M_1+M_2+\ldots+M_n), \qquad \ldots(7)$$

where $M_1, M_2, \ldots M_n$ are the imaginary parts of n pairs of conjugate constants of partial fractions corresponding to n pairs of conjugate factors of $F(x)=0$.

We shall use the result (7) in the following articles.

An important rule for partial fractions.

Split into partial fractions $\dfrac{x-1}{(x-2)(x-3)} = \dfrac{f(x)}{F(x)}$,

where $F(x)=x^2-5x+6$, *i.e.* $F'(x)=2x-5$,

$$\therefore \quad F'(2)=-1,\ F'(3)=1.$$

Now by method of suppression the partial fractions are

$$\frac{2-1}{(x-2)(2-3)}+\frac{3-1}{(x-3)(3-2)},$$

i.e.

$$\frac{f(2)}{F'(2)(x-2)}+\frac{f(3)}{F'(3)(x-3)}.$$

Hence if α be a root of $F(x) = 0$, then the partial fraction corresponding to factor $(x-\alpha)$ of the denominator of $\dfrac{f(x)}{F(x)}$ is

$$\frac{f(\alpha)}{F'(\alpha)(x-\alpha)}.$$

§ 5. To find the value of integral $\displaystyle\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$, where m and n are positive integers and $n > m$

(Agra 46, 60 ; Rajputana 57)

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx = 2\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx \qquad \text{(by prop. V)}.$$

$$\therefore \quad \int_0^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^{2m}}{1+x^{2n}}\, dx$$

$$= \tfrac{1}{2}\, 2\pi\,[M_1+M_2+\ldots+M_n]$$

$$= \pi\,[M_1+M_2+\ldots+M_n], \ldots(1)$$

where $M_1, M_2, \ldots M_n$ are the imaginary parts of the constants of partial fractions.

Let $a+ib=\alpha$ say be a root of the equation $x^{2n}+1=0$, so that the corresponding factor is $x-\alpha=(x-a)-ib$ and the corresponding partial fraction is $\dfrac{L-iM}{(x-a)-ib}$ and we have to find the value of M.

Now as discussed in important note, the partial fraction of $\dfrac{f(x)}{F(x)}$, corresponding to a factor $x-\alpha$ is $\dfrac{f(\alpha)}{F'(\alpha)\,(x-\alpha)}$.

Hence $L-iM$ corresponding to root α is $\dfrac{f(\alpha)}{F'(\alpha)}$.

$$\therefore \quad L-iM=\frac{\alpha^{2m}}{2n.\alpha^{2n-1}}=\frac{\alpha^{2m+1}}{2n.\alpha^{2n}}.$$

But α is a root of $x^{2n}+1=0$; $\therefore$ $\alpha^{2n}+1=0$ or $\alpha^{2n}=-1$.

$$\therefore \quad L-iM=\frac{\alpha^{2m+1}}{-2n}. \qquad \ldots(2)$$

Now we should find α, where α is a root of $1+x^{2n}=0$.

$\therefore$ $x^{2n}=-1=\cos\pi\pm i\sin\pi=\cos(2r\pi+\pi)\pm i\sin(2r\pi+\pi)$.

$$\therefore \quad x=[\cos(2r+1)\pi\pm i\sin(2r+1)\pi]^{1/2n}$$

$$=\cos\frac{(2r+1)\pi}{2n}\pm i\sin\frac{(2r+1)\pi}{2n},$$

where $r=0, 1, 2, \ldots(n-1)$, each value of r giving rise to two values of x

or $$x=\cos\frac{(2r+1)\pi}{2n}\pm i\sin\frac{(2r+1)\pi}{2n}.$$

$$\therefore \quad \alpha=a+ib=\cos\frac{(2r+1)\pi}{2n}\pm i\sin\frac{(2r+1)\pi}{2n}.$$

$$\therefore \quad \alpha^{2m+1}=\cos\frac{(2r+1)}{2n}(2m+1)\pi+i\sin\frac{(2r+1)}{2n}(2m+1)\pi$$

or $\alpha^{2m+1}=\cos(2r+1)\theta+i\sin(2r+1)\theta$, where $\theta=\dfrac{2m+1}{2n}\pi$ and $r=0, 1, 2, \ldots(n-1)$.

Hence from (2), $L-iM=\dfrac{x^{2m+1}}{-2n}$

or $$L-iM=-\frac{1}{2n}\{\cos(2r+1)\,\theta+i\sin(2r+1)\,\theta\}.$$

Equating imaginary parts, we get

$$M=\frac{1}{2n}\sin(2r+1)\,\theta,\ \text{ where } r=0,\ 1,\ 2,\ \ldots(n-1).$$

Let the values of M, corresponding to values of r be denoted by $M_1, M_2, M_3, \ldots M_n$.

$\therefore\ M_1+M_2+M_3+\ldots+M_n$

$$=\frac{1}{2n}[\sin\theta+\sin 3\theta+\sin 5\theta+\ldots\sin(2n-1)\,\theta]. \quad \ldots(3)$$

Now we know from author's Trigonometry that sum of the sines of angles (angles being in A.P.)

$$=\frac{\sin n\ \dfrac{\text{diff}}{2}}{\sin\dfrac{\text{diff.}}{2}}\sin\left(\frac{\text{1st angle}+\text{last angle}}{2}\right).$$

$\therefore\ M_1+M_2+\ldots+M_n$

$$=\frac{1}{2n}\cdot\frac{\sin n.\dfrac{2\theta}{2}}{\sin\dfrac{2\theta}{2}}\sin\left[\frac{\theta+(2n-1)\,\theta}{2}\right]$$

$$=\frac{1}{2n}\frac{\sin^2 n\theta}{\sin\theta}=\frac{1}{2n\sin\theta}\sin^2\frac{(2m+1)}{2}\pi,$$

$$\because\ \theta=\frac{(2m+1)}{2n}\pi$$

or $$\Sigma M_1=\frac{1}{2n\sin\theta}\sin^2\left(m\pi+\frac{\pi}{2}\right)=\frac{1}{2n\sin\theta}(\pm 1)^2$$

$$=\frac{1}{2n\sin\theta}. \quad \ldots(4)$$

Hence from (1) by the help of (4).

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx=\pi\Sigma M_1=\frac{\pi}{2n\sin\theta}=\frac{\pi}{2n\sin\frac{(2m+1)}{2n}\pi}$$

or
$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx=\frac{\pi}{2n}\operatorname{cosec}\left(\frac{2m+1}{2n}\pi\right). \quad \ldots(5)$$

§ 6. To find the value of $\int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx$, where m, n are +ive integers and n > m.

The equation $1-x^{2n}=0$ has two real roots $+1$ and -1 and the rest $(2n-2)$ are all imaginary in $(n-1)$ conjugate pairs. The partial fractions corresponding to $+1$ and -1 according to rule $\frac{f(\alpha)}{F'(\alpha)(x-\alpha)}$ are

$$\frac{(-1)^{2m}}{-2n(-1)^{2n-1}(x+1)}+\frac{(1)^{2m}}{-2n(1)^{2n-1}(x-1)}=\frac{1}{2n}\left(\frac{1}{x+1}-\frac{1}{x-1}\right)$$
$$=\frac{1}{n(1-x^4)}.$$

We shall show that $\int_0^\infty \frac{1}{(1-x^4)}\,dx=0.$

$$\int_0^\infty \frac{dx}{1-x^2}=\int_0^1 \frac{dx}{1-x^2}+\int_1^\infty \frac{dx}{1-x^2}.$$

Put $x=\frac{1}{z}$ in the 2nd integral ; $\therefore\ dx=-\frac{1}{z^2}\,dz.$

$$\therefore\ \int_1^\infty \frac{dx}{1-x^2}=\int_1^0 \frac{-1/z^2\,dr}{(1-1/z^2)}=\int_1^0 \frac{dz}{1-z^2}=-\int_0^1 \frac{dx}{1-x^2}$$

or
$$\int_0^\infty \frac{dx}{1-x^2}=\int_0^1 \frac{dx}{1-x^2}-\int_0^1 \frac{dx}{1-x^2}=0.$$

Hence we show that the part of definite integral corresponding to real roots of $1-x^{2n}=0$ vanishes and we have now to consider the integral corresponding to $(n-1)$ conjugate pairs of imaginary roots.

being real corresponding to $r=0$ and n, *i.e.* $x=\pm 1$.

$$\therefore \quad \alpha=a+ib=\cos\frac{r\pi}{n}+\ \sin\frac{r\pi}{n}.$$

$$\therefore \quad \alpha^{2m+1}=\cos\frac{r\pi}{n}(2m+1)+i\sin\frac{r\pi}{n}(2m+1)$$

$$=\cos r\theta+\sin r\theta, \quad \text{where } \theta=\frac{(2m+1)}{n}\pi.$$

$$\therefore \quad L-iM=\frac{\alpha^{2m+1}}{-2n} \text{ by (2)}$$

or $\quad L-iM=-\dfrac{1}{2n}(\cos r\theta+i\sin r\theta),\ r=1, 2, \ldots (n-1).$

Equating imaginary parts, we get $M=\dfrac{1}{2n}\sin r\theta$.

Let the values of M corresponding to $(n-1)$ values of r be denoted by $M_1, M_2 \ldots M_{n-1}$.

$$\therefore \quad M_1+M_2+M_{n-1}$$

$$=\frac{1}{2n}[\sin\theta+\sin 2\theta+\ldots+\sin(n-1]$$

$$=\frac{1}{2n}\cdot\frac{\sin(n-1)\theta/2}{\sin\theta/2}\sin\left[\frac{\theta+(n-1)\theta}{2}\right],$$

as there are $(n-1)$ terms

or $\quad \Sigma M_1=\dfrac{1}{2n\sin\theta/2}\cdot\sin(n-1)\theta/2\sin n\theta/2. \qquad \ldots(2)$

Now $\sin(n-1)\frac{1}{2}\theta.\sin n\theta/2$

$$=\left(\sin\frac{n\theta}{2}\cos\frac{\theta}{2}-\cos\frac{n\theta}{2}\sin\frac{\theta}{2}\right)\sin\frac{n\theta}{2}$$

$$=\left(\sin^2\frac{n\theta}{2}\cos\frac{\theta}{2}-\frac{1}{2}\sin n\theta.\sin\frac{\theta}{2}\right). \qquad \text{Put } \theta=\frac{(2m+1)}{n}\pi.$$

$$\therefore \quad n\theta=(2m+1)\pi \text{ or } \sin n\theta=\sin(2m+1)\pi=0$$

and $\quad \sin^2\dfrac{n\theta}{2}=\sin^2(2m+1)\dfrac{\pi}{2}=\sin^2\left(m\pi+\dfrac{\pi}{2}\right)=(\pm 1)^2=1.$

$$\int_{-\infty}^{\infty} \frac{x^{2m}}{1-x^{2n}}\,dx=2\pi\,(M_1+M_2+\dots M_{n-1})$$

where $M_1, M_2, \dots M_{n-1}$ are the imaginary parts of the constants of partial fractions

or $$2\int_0^{\infty} \frac{x^{2m}}{1-x^{2n}}\,dx=2\pi\,(M_1+M_2+\dots M_{n-1}).$$

$$\therefore\quad \int_0^{\infty} \frac{x^{2m}}{1-x^{2n}}\,dx=\pi\,(M_1+M_2+\dots M_{n-1}). \qquad \dots(1)$$

Let $a+ib=\alpha$ say be the imaginary root of $1-x^{2n}=0$, so that the corresponding factor is $x-\alpha=(x-a)-ib$ and the corresponding partial fraction be $\dfrac{L-iM}{(x-a)-ib}$ and we have to find the value of M.

Now as discussed in important note P. 58 the partial fraction corresponding to factor $(x-\alpha)$ of

$$\frac{f(x)}{F(x)} \text{ is } \frac{f(\alpha)}{F'(\alpha)\,(x-\alpha)}.$$

Hence $L-iM$ corresponding to a root α is $\dfrac{f(\alpha)}{F'(\alpha)}$.

$$\therefore\quad L-iM=\frac{\alpha^{2m}}{-2n\alpha^{2n-1}}=\frac{\alpha^{2m+1}}{-2n\alpha^{2n}}.$$

But α is imaginary root of $1-x^{2n}=0$; $\therefore\ \alpha^{2n}=1$.

$$\therefore\quad L-iM=\frac{\alpha^{2m+1}}{-2n}.$$

Now we should find α, where α is the imaginary root of $1-x^{2n}=0$.

$$x^{2n}=1=\cos 0\pm i\sin 0=(\cos 2r\pi\pm i\sin 2r\pi).$$

$$\therefore\quad x=(\cos 2r\pi\pm i\sin 2r\pi)^{1/2n}=\cos\left(\frac{r\pi}{n}\pm i\sin\frac{r\pi}{n}\right).$$

where $r=1, 2\dots(n-1)$.

each value of r giving rise to two roots and thus $(n-1)$ values of r will give us $(2n-2)$ imaginary roots and the rest two

being real corresponding to $r=0$ and n, *i.e.* $x=\pm 1$.

$$\therefore \quad \alpha=a+ib=\cos\frac{r\pi}{n}+\ \sin\frac{r\pi}{n}.$$

$$\therefore \quad \alpha^{2m+1}=\cos\frac{r\pi}{n}(2m+1)+i\sin\frac{r\pi}{n}(2m+1)$$

$$=\cos r\theta+\ \sin r\theta,\quad \text{where } \theta=\frac{(2m+1)}{n}\pi.$$

$$\therefore \quad L-iM=\frac{\alpha^{2m+1}}{-2n} \text{ by (2)}$$

or $\quad L-iM=-\dfrac{1}{2n}(\cos r\theta+i\sin r\theta),\ r=1, 2, \ldots (n-1)$

Equating imaginary parts, we get $M=\dfrac{1}{2n}\sin r\theta$.

Let the values of M corresponding to $(n-1)$ values of r be denoted by $M_1, M_2 \ldots M_{n-1}$.

$$\therefore \quad M_1+M_2+M_{n-1}$$

$$=\frac{1}{2n}[\sin\theta+\sin 2\theta+\ldots+\sin(n-1]$$

$$=\frac{1}{2n}\cdot\frac{\sin(n-1)\theta/2}{\sin\theta/2}\sin\left[\frac{\theta+(n-1)\theta}{2}\right],$$

as there are $(n-1)$ terms

or $\quad \Sigma M_1=\dfrac{1}{2n\sin\theta/2}\cdot\sin(n-1)\theta/2\sin n\theta/2. \quad \ldots(2)$

Now $\sin(n-1)\frac{1}{2}\theta\cdot\sin n\theta/2$

$$=\left(\sin\frac{n\theta}{2}\cos\frac{\theta}{2}-\cos\frac{n\theta}{2}\sin\frac{\theta}{2}\right)\sin\frac{n\theta}{2}$$

$$=\left(\sin^2\frac{n\theta}{2}\cos\frac{\theta}{2}-\frac{1}{2}\sin n\theta\cdot\sin\frac{\theta}{2}\right). \qquad \text{Put } \theta=\frac{(2m+1)}{n}\pi.$$

$$\therefore \quad n\theta=(2m+1)\pi \text{ or } \sin n\theta=\sin(2m+1)\pi=0$$

and $\quad \sin^2\dfrac{n\theta}{2}=\sin^2(2m+1)\dfrac{\pi}{2}=\sin^2\left(m\pi+\dfrac{\pi}{2}\right)=(\pm 1)^2=1.$

$$\therefore \quad M_1+M_2+\ldots+M_{n-1}=\frac{1}{2n \sin \theta/2}\left(1.\cos\frac{\theta}{2}-0\right)$$
by (2)

$$=\frac{1}{2n}\cot\frac{\theta}{2}. \qquad \ldots(4)$$

Hence from (1) by the help of (4), we have

$$\int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx=\pi\,(M_1+M_2+\ldots M_{n-1})=\frac{\pi}{2n}\cot\frac{\theta}{2}$$

or
$$\int_0^\infty \frac{\mathbf{x}^{2m}}{1-\mathbf{x}^{2n}}\,d\mathbf{x}=\frac{\pi}{2n}\cot\frac{(2m+1)}{2n}\pi.$$

§ 7. Some important deductions.

We have proved the following definite integrals :

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx=\frac{\pi}{2n}\operatorname{cosec}\frac{2m+1}{2n}\pi \qquad \ldots(1)$$

and
$$\int_0^\infty \frac{x^{2m}}{1-x^{2n}}\,dx=\frac{\pi}{2n}\cot\frac{2m+1}{2n}\pi. \qquad \ldots(2)$$

By making proper substitutions we shall deduce certain standard results from (1) and (2).

(a) Put $x^{2n}=z$ and $\frac{2m+1}{2n}=a$, where a is less than unity.

$$\therefore \quad 2nx^{2n-1}\,dx=dz.$$

Also
$$a-1=\frac{2m+1}{2n}-1=\frac{2m+1-2n}{2n}.$$

$$\therefore \quad \int_0^\infty \frac{x^{2m}}{1+z}\cdot\frac{dz}{2nx^{2n-1}}=\frac{\pi}{2n}\operatorname{cosec} a\pi$$

or
$$\int_0^\infty \frac{x^{2m+1-2n}}{1+z}\,dz=\pi\operatorname{cosec} a\pi \text{ or } \int_0^\infty \frac{[x^{2n}]^{(2m+1-2n)/2n}}{1+z}\,dz$$
$$=\pi\operatorname{cosec} a\pi$$

or
$$\int_0^\infty \frac{\mathbf{z}^{a-1}}{1+\mathbf{z}}\,d\mathbf{z}=\pi\operatorname{cosec} a\pi. \qquad \ldots(3)$$

Similarly from (2) by the same substitution, we get

$$\int_0^\infty \frac{\mathbf{z}^{a-1}}{1-\mathbf{z}}\,d\mathbf{z}=\pi\cot a\pi. \qquad \ldots(4)$$

(b) Again putting $z^a = u$ in (3) and (4), we get $az^{a-1}\,dz = du$ and adjust the limits

$$\therefore \quad \frac{1}{a}\int_0^\infty \frac{du}{1+u^{1/a}} = \pi \text{ cosec } a\pi \text{ and } \frac{1}{a}\int_0^\infty \frac{du}{1-u^{1/a}} = \pi \cot a\pi$$

or
$$\int_0^\infty \frac{du}{1+u^{1/a}} = a\pi \text{ cosec } a\pi \qquad \ldots(5)$$

and
$$\int_0^\infty \frac{du}{1-u^{1/a}} = a\pi \cot a\pi. \qquad \ldots(6)$$

Another form of (5) and (6).

Let us put $\frac{1}{a} = r$ and as $a < 1$, $\therefore r > 1$

Then
$$\int_0^\infty \frac{du}{1+u^r} = \frac{\pi}{r} \text{ cosec } \frac{\pi}{r} \qquad \ldots(7)$$

and
$$\int_0^\infty \frac{du}{1-u^r} = \frac{\pi}{r} \cot \frac{\pi}{r}. \qquad \ldots(8)$$

§ 8. To prove that

$$\int_0^1 \frac{x^n + x^{-n}}{x + x^{-1}} \frac{dx}{x} = \frac{\pi}{2} \sec \frac{n\pi}{2} = \int_0^\infty \frac{x^n}{x^2+1}\,dx.$$

$$\int_0^1 \frac{x^n - x^{-n}}{x - x^{-1}} \frac{dx}{x} = \frac{\pi}{2} \tan \frac{n\pi}{2} = \int_0^\infty \frac{x^n}{x^2-1}\,dx.$$

We know that

$$\int_0^\infty \frac{x^n}{x^2+1}\,dx = \int_0^1 \frac{x^n}{x^2+1}\,dx + \int_1^\infty \frac{x^n}{x^2+1}\,dx. \qquad \ldots(1)$$

Put $x = \frac{1}{z}$ in the second and adjust the limits.

$$\therefore \quad \int_1^\infty \frac{x^n}{x^2+1}\,dx = \int_1^0 \frac{z^{-n}}{(1+z^2)} z^2 . \left(-\frac{1}{z^2}\,dz\right)$$
$$= \int_0^1 \frac{z^{-n}}{(1+z^2)}\,dz = \int_0^1 \frac{x^{-n}}{1+x^2}\,dx. \quad \ldots(2)$$

Hence from (1) by the help of (2), we have

$$\int_0^\infty \frac{x^n}{x^2+1}\,dx=\int_0^1 \frac{x^n}{x^2+1}\,dx+\int_0^1 \frac{x^{-n}}{x^2+1}\,dx$$

$$=\int_0^1 \frac{x^n+x^{-n}}{(x+x^{-1})}\cdot\frac{dx}{x}. \quad \ldots(3)$$

Now put $x^2=z$ in L. H. S. integral ; $\therefore 2x\,dx=dz$.

$$\therefore \quad \int_0^\infty \frac{x^n}{x^2+1}\,dx=\int_0^\infty \frac{x^{n-1}}{1+z}\cdot\frac{dz}{2}=\int_0^\infty \frac{(x^2)^{(n-1)/2}}{1+z}\cdot\frac{dz}{2}$$

$$=\frac{1}{2}\int_0^\infty \frac{z^{\frac{1}{2}(n+1)-1}}{1+z},$$

which is of the form

$$\int_0^\infty \frac{z^{a-1}}{1+z}\,dz=\pi \operatorname{cosec} a\pi.$$

$$\therefore \quad \int_0^\infty \frac{x^n}{x^2+1}\,dx=\frac{1}{2}\pi \operatorname{cosec}\frac{n+1}{2}\pi=\frac{\pi}{2}\operatorname{cosec}\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)$$

$$=\frac{\pi}{2}\sec\frac{n\pi}{2}. \quad \ldots(4)$$

Hence from (3) and (4), we get

$$\int_0^1 \frac{x^n+x^{-n}}{(x+x^{-1})}\frac{dx}{x}=\frac{\pi}{2}\sec\frac{n\pi}{2}=\int_0^\infty \frac{x^n}{x^2+1}\,dx. \quad \ldots(5)$$

Similarly proceeding as above, we can show that

$$\int_0^\infty \frac{x^n}{x^2-1}\,dx=\int_0^1 \frac{x^n\,dx}{x^2-1}-\int_0^1 \frac{x^{-n}}{x^2-1}\,dx=\int_0^1 \frac{x^n-x^{-n}}{x-x^{-1}}\frac{dx}{x}.$$

Also on putting $x^2=z$ in L.H.S. it will reduce to form

$$-\frac{1}{2}\int_0^\infty \frac{z^{\frac{1}{2}(n+1)-1}}{1-z}\,dz.$$

$$\int_0^\infty \frac{z^{a-1}}{1-z}\,dz=-\pi\cot a\pi \text{ and here } a=\frac{n+1}{2}.$$

$$\therefore \quad \int_0^\infty \frac{x^n-x^{-n}}{(x-x^{-1})}\frac{dx}{x}=-\frac{1}{2}\pi\cot\frac{n+1}{2}\pi$$

$$=-\frac{\pi}{2}\cot\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)=\frac{\pi}{2}\tan\frac{n\pi}{2}.$$

$$\therefore \quad \int_0^\infty \frac{x^n-x^{-n}}{x-x^{-1}}\frac{dx}{x}=\frac{\pi}{2}\tan\frac{n\pi}{2}=\int_0^\infty \frac{x^n}{x^2-1}\,dx. \quad \ldots($$

Deductions.

Putting $x = e^{-z}$ in (5) and (6), so that

$$dx = -e^{-z}\,dz = -x\,dz \text{ or } \frac{dx}{x} = -dz.$$

Also when $x = \infty$, $z = -\infty$; when $x = 0$, $z = \infty$.

$$\int_0^\infty \frac{x^n + x^{-n}}{x + x^{-1}} \frac{dx}{x} = \frac{\pi}{2} \sec \frac{n\pi}{2}$$

$$\therefore \int_\infty^{-\infty} \frac{e^{-nz} + e^{nz}}{e^{-z} + e^{z}} (-dz) = \frac{\pi}{2} \sec \frac{n\pi}{2}$$

Now put $n\pi = a$

$$\therefore \int_0^\infty \frac{e^{az} + e^{-az}}{e^{\pi z} + e^{-\pi z}}\, dz = \frac{1}{2} \sec \frac{a}{2}. \quad \ldots(7)$$

Again we know that

$$\int_0^\infty \frac{x^n - x^{-n}}{x - x^{-1}} \frac{dx}{x} = \frac{\pi}{2} \tan \frac{n\pi}{2}.$$

Proceeding as above, we have

$$\int_\infty^{-\infty} \frac{e^{-nz} - e^{nz}}{e^{-z} - e^{z}} (-dz) = \frac{\pi}{2} \tan \frac{n\pi}{2}.$$

Putting $n\pi = a$, we get

$$\int_0^\infty \frac{e^{az} - e^{-az}}{e^{\pi z} - e^{-\pi z}}\, dz = \frac{1}{2} \tan \frac{a}{2}. \quad \ldots(8)$$

Note. *Before doing questions to follow, students should make sure that they remember the results of all the formulae and deductions proved so far on P. 50, 64 and 67.*

Exercises

Ex. 1. *If n lies between -1 and $+1$, show that*

$$\int_0^{\pi/2} \tan^n \theta\, d\theta = \frac{\pi}{2 \cos \frac{n\pi}{2}}.$$

(Agra 45, 49)

Put $\tan \theta = x$; $\sec^2 \theta\, d\theta = dx$ or $d\theta = \dfrac{dx}{1+\tan^2\theta} = \dfrac{dx}{1+x^2}$.

$$I=\int_0^\infty \frac{x^n}{1+x^2}\,dx=\frac{\pi}{2}\sec\frac{n\pi}{2}=\frac{\pi}{2\cos\frac{n\pi}{2}}.\quad \text{(R. 5 P. 66)}$$

Ex. 2. $\int_0^\infty \frac{dx}{1-x^4}=\frac{\pi}{4}$

We know that $\int_0^\infty \frac{x^{2m}\,dx}{1-x^{2n}}=\frac{\pi}{2n}\cot\frac{(2m+1)}{2n}\pi.$ (P. 64)

Put $m=0$ and $n=2$ in the above

$$\therefore\quad \int_0^\infty \frac{dx}{1-x^4}=\frac{\pi}{4}\cot\frac{\pi}{4}=\frac{\pi}{4}$$

Ex. 3. $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}=\frac{\pi}{n}\,cosec\,\frac{\pi}{n}$

$$I=\int_0^1 \frac{dx}{x\,[(1/x^n)-1]^{1/n}}$$

Put $\frac{1}{x^n}-1=\frac{1}{y^n}$. $\therefore\quad -\frac{n}{x^{n+1}}\,dx=-\frac{n}{y^{n+1}}\,dy.$

Also when $x=1$, then $\frac{1}{y^n}=0$, $\therefore\ y=\infty$ and when $x=0$.

then $$\frac{1}{y^n}=\infty\ ;\quad \therefore\quad y=0$$

$$\therefore\quad I=\int_0^\infty \frac{1}{x}\cdot\frac{1}{1/y}\cdot\frac{x^{n+1}}{y^{n+1}}\,dy=\int_0^\infty \frac{x^n\,dy}{y^n}.$$

But $\frac{1}{x^n}-1=\frac{1}{y^n}$, $\therefore\quad \frac{y^n}{x^n}-y^n=1$ or $\frac{y^n}{x^n}=1+y^n$.

$$\therefore\quad I=\int_0^\infty \frac{dy}{1+y^n}.$$

Again we know that

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx=\frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)}{2n}\pi.$$

Putting $m=0$ and $n=n/2$, we get

$$\int_0^\infty \frac{dx}{1+x^n}=\int_0^\infty \frac{dy}{1+y^n}=\frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}.$$

(b) $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{\pi}{n} \operatorname{cosec} \frac{\pi}{n}$.

It is exactly as part (a). $I = \int_0^a \frac{dx}{x\,[(a/x)^n - 1]^{1/n}}$.

Now put $\left(\frac{a}{x}\right)^n - 1 = \frac{1}{y^n}$ etc.

Ex. 4. *Prove that*

$$\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}\, dx = \frac{1}{(1+c)^n} \cdot \frac{\pi}{\sin n\pi},$$

where $0 < n < 1$ **(Agra 51)**

Put $\frac{x}{1-x} = z$; $\because$ $x = \frac{z}{1+z} = 1 - \frac{1}{1+z}$.

$$\therefore \quad dx = \frac{1}{(1+z)^2}\, dz.$$

Also when $x=0$, $z=0$ and when $x=1$, then $z=\infty$.

$$\therefore \quad I = \int_0^\infty \frac{z^{n-1}}{(1+z)^{n-1}} \cdot \frac{1}{\left(1+\frac{cz}{1+z}\right)\left(1-\frac{z}{1+z}\right)^n} \cdot \frac{1}{(1+z)^2}\, dz$$

$$= \int_0^\infty \frac{z^{n-1}}{[1+z(1+c)]}\, dz.$$

Again put $z(1+c) = t$; $\therefore$ $(1+c)\, dz = dt$.

$$\therefore \quad I = \frac{1}{(1+c)^{n-1}} \int_0^\infty \frac{t^{n-1}}{1+t} \cdot \frac{dt}{(1+c)} = \frac{1}{(1+c)^n} \int_0^\infty \frac{t^{n-1}}{1+t}\, dt.$$

Now we know that $\int_0^\infty \frac{z^{a-1}}{1+z} = \frac{\pi}{\sin a\pi}$, where $a < 1$.

[Result 3 P. 64]

$$\therefore \quad I = \frac{1}{(1+c)^n} \cdot \frac{\pi}{\sin n\pi}, \text{ where } n < 1.$$

Ex. 5. *Prove that* $\int_0^1 \frac{z^a - z^{-a}}{1-z}\, dz = \pi \cot a\pi - \frac{1}{a}$.

(Sagar 62 ; Vikram 63 ; Rajputana 63, 49)

$$I=\int_0^1 \frac{z^a}{1-z}\,dz-\int_0^1 \frac{z^{-a}}{1-z}\,dz. \qquad \ldots(1)$$

Now $\displaystyle\int_0^1 \frac{z^a}{1-z}\,dz=-\int_0^1 \frac{z^{a-1}(-z)}{1-z}\,dz$

$$=-\int_0^1 \frac{z^{a-1}(1-z-1)}{1-z}\,dz=-\int_0^1 z^{a-1}\,dz+\int_0^1 \frac{z^{a-1}}{1-z}\,dz$$

$$=-\left[\frac{z^a}{a}\right]_0^1+\int_0^1 \frac{z^{a-1}}{1-z}\,dz=-\frac{1}{a}+\int_0^1 \frac{z^{a-1}}{1-z}\,dz.$$

$$\therefore \quad I=-\frac{1}{a}+\int_0^1 \frac{z^{a-1}}{1-z}\,dz-\int_0^1 \frac{z^{-a}}{1-z}\,dz \text{ by (1).} \qquad \ldots(2)$$

Put $z=\dfrac{1}{x}$ in the last integral.

$$\therefore \quad \int_0^1 \frac{z^{-a}}{1-z}\,dz=\int_\infty^1 \frac{x^a}{\left(1-\frac{1}{x}\right)}\left(-\frac{1}{x^2}\,dx\right)=-\int_1^\infty \frac{x^{a-1}}{1-x}\,dx$$

$$=-\int_1^\infty \frac{z^{a-1}}{1-z}\,dz. \qquad \ldots(3)$$

Hence from (2) by the help of (3), we get

$$I=-\frac{1}{a}+\int_0^1 \frac{z^{a-1}}{1-z}\,dz+\int_1^\infty \frac{z^{a-1}}{1-z}\,dz$$

$$=-\frac{1}{a}+\int_0^\infty \frac{z^{a-1}}{1-z}\,dz=-\frac{1}{a}+\pi\cot a\pi. \qquad \textbf{(R. 4 P. 64)}$$

Ex. 6. *Evaluate* $\displaystyle\int_0^1 \frac{x^m+x^{-m}}{x^n+x^{-n}}\cdot\frac{dx}{x}$, *where* $n > m$.

The form of the given integral gives us the idea to reduce it to the form

$$\int_0^1 \frac{x^n+x^{-n}}{x+x^{-1}}\,\frac{dx}{x}=\frac{\pi}{2}\sec\frac{n\pi}{2}=\int_0^\infty \frac{x^n}{1+x^2}\,dx. \quad \text{[§ 8 P. 65]}$$

Put $x^n=z$ in the given integral, so that

$$nx^{n-1}\,dx=dz\,; \quad \therefore \quad nx^n\frac{dx}{x}=dz \quad \text{or} \quad \frac{dx}{x}=\frac{dz}{nz}$$

Also $\quad x = z^{1/n} \quad$ or $\quad x^m = z^{m/n}$.

$$\therefore \quad I = \frac{1}{n} \int_0^1 \frac{z^{m/n} + z^{-m/n}}{z + z^{-1}} \frac{dz}{z} = \frac{1}{n} \cdot \frac{\pi}{2} \sec\left(\frac{m}{n} \cdot \frac{\pi}{2}\right).$$

Ex. 7. *Evaluate* $\displaystyle\int_0^\infty \frac{(e^{ax}+e^{-ax})(e^{bx}-e^{-bx})}{e^{\pi x}-e^{-\pi x}} \, dx.$

The form of the given integral gives us the idea to educe it to the forms

$$\int_0^\infty \frac{e^{az}+e^{-az}}{e^{\pi z}+e^{-\pi z}} \, dz = \tfrac{1}{2} \sec \frac{a}{2} \text{ and } \int_0^\infty \frac{e^{az}-e^{-az}}{e^{\pi z}-e^{-\pi z}} \, dz = \tfrac{1}{2} \tan \frac{a}{2}.$$

The numerator of given integral by actual multiplication is

$$\int_0^\infty \frac{[e^{(a+b)x} - e^{-(a+b)x}]}{e^{\pi x}-e^{-\pi x}} \, dx - \int_0^\infty \frac{[e^{(a-b)x} - e^{-(a-b)x}]}{e^{\pi x}-e^{-\pi x}} \, dx$$

$$= \tfrac{1}{2} \tan \frac{a+b}{2} - \tfrac{1}{2} \tan \frac{a-b}{2} \qquad \textbf{(R. 8 P. 67)}$$

$$= \frac{1}{2} \frac{\sin\left(\frac{a+b}{2} - \frac{a-b}{2}\right)}{\cos \frac{a+b}{2} \cos \frac{a-b}{2}} = \frac{\sin b}{\cos a + \cos b}.$$

Ex. 8. *Prove* $\displaystyle\int_0^\infty \frac{(e^{ax}+e^{-ax})(e^{bx}+e^{-bx})}{e^{\pi x}+e^{-\pi x}} = \frac{2 \cos \frac{a}{2} \cos \frac{b}{2}}{\cos a + \cos b}.$

Proceding exactly as above and using 1st formula given in Ex. 7,

$$I = \frac{1}{2} \left\{ \frac{1}{\cos \frac{a+b}{2}} + \frac{1}{\cos \frac{a-b}{2}} \right\} \qquad \textbf{(R. 7 P. 67)}$$

$$= \frac{\cos \frac{a-b}{2} + \cos \frac{a+b}{2}}{2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}} = \frac{2 \cos \frac{a}{2} \cos \frac{b}{2}}{\cos a + \cos b}.$$

Ex. 9. *Evaluate* $\displaystyle\int_0^\infty \frac{e^{bx}+e^{-bx}}{e^{\pi x}+e^{-\pi x}} \sin ax \, dx.$

We know that $\sin ax = \dfrac{e^{iax} - e^{-iax}}{2i}$

$$\therefore \quad I = \frac{1}{2i} \int_0^\infty \frac{(e^{bx} + e^{-bx})(e^{iax} - e^{-iax})}{e^{\pi x} - e^{-\pi x}}.$$

Proceeding as in Ex 7

$$I = \frac{1}{2i} \left[\int_0^\infty \frac{e^{(b+ia)x} - e^{-(b+ia)x}}{e^{\pi x} - e^{-\pi x}} \, dx - \int_0^\infty \frac{e^{(b-ia)x} - e^{-(b-ia)x}}{e^{\pi x} - e^{-\pi x}} \, dx \right]$$

$$= \frac{1}{2i} \cdot \frac{1}{2} \left[\tan \frac{b+ia}{2} - \tan \frac{b-ia}{2} \right]$$

$$= \frac{1}{2i} \frac{\sin \left(\frac{b+ia}{2} - \frac{b-ia}{2} \right)}{2 \cos \frac{b+ia}{2} \cos \frac{b-ia}{2}}$$

$$= \frac{1}{2i} \frac{\sin ia}{\cos b + \cos ia} = \frac{1}{2} \frac{\sinh a}{\cos b + \cosh a}$$

$\because \ \sin ia = i \sinh a$ and $\cos ia = \cosh a$.

Ex. 10. *Evaluate* $\displaystyle\int_0^\infty \frac{e^{bx} - e^{-bx}}{e^{\pi x} - e^{-\pi x}} \cos ax \, dx.$

$$I = \frac{1}{2} \int_0^\infty \frac{(e^{bx} - e^{-bx})(e^{iax} + e^{-iax})}{(e^{\pi x} - e^{-\pi x})} \, dx$$

$$= \frac{1}{2} \left[\int_0^\infty \frac{e^{(b+ia)x} - e^{-(b+ia)x}}{e^{\pi x} - e^{-\pi x}} \, dx + \int_0^\infty \frac{e^{(b-ia)x} - e^{-(b-ia)x}}{e^{\pi x} - e^{-\pi x}} \, dx \right]$$

$$= \frac{1}{2} \left[\frac{1}{2} \tan \frac{b+ia}{2} + \frac{1}{2} \tan \frac{b-ia}{2} \right] = \frac{1}{4} \frac{\sin \left(\frac{b+ia}{2} + \frac{b-ia}{2} \right)}{\cos \frac{b+ia}{2} \cos \frac{b-ia}{2}}$$

$$= \frac{1}{2} \frac{\sin b}{\cos b + \cos ia} = \frac{\sin b}{2 \cos b + 2 \cosh a} = \frac{\sin b}{2 \cdots + e^{-a}}$$

$\tan A - \tan B = \dfrac{\sin A \cos B \ldots}{\cos \ldots}$ [illegible]

Ex. 11. *Evaluate* $\int_0^{\infty} \frac{e^{bx}+e^{-bx}}{e^{\pi x}+e^{-\pi x}} \cos ax\, dx$

Proceeding exactly as above

$$I=\left[\frac{1}{2}\sec\frac{b+ia}{2}+\frac{1}{2}\sec\frac{b-ia}{2}\right]$$

$$=\frac{\cos\frac{b-ia}{2}+\cos\frac{b+ia}{2}}{2\cos\frac{b+ia}{2}\cos\frac{b-ia}{2}}=\frac{1}{2}\cdot\frac{2\cos(b/2)\cos ia/2}{\cos b+\cos ia}$$

$$=\frac{\cos b/2\cosh a/2}{\cos b+\cosh a}=\frac{\cos(b/2)\,(e^{a/2}+e^{-a/2})}{2\cos b+e^{a}+e^{-a}}$$

Ex. 12. *Prove that* $\int_0^{\infty} \frac{\sin a\theta}{e^{\pi\theta}-e^{-\pi\theta}}=\frac{1}{4}\frac{e^{a}-1}{e^{a}+1}$

Proceeding as above,

$$I=\frac{1}{2i}\cdot\frac{1}{2}\tan\frac{ia}{2}=\frac{1}{4i}\,i\tanh\frac{a}{2}$$

$$=\frac{1}{4}\frac{e^{a/2}-e^{-a/2}}{e^{a/2}+e^{-a/2}}=\frac{1}{4}\frac{e^{a}-1}{e^{a}+1}.$$

Ex. 13. *Prove that*

$$\int_0^{\infty}\frac{\cosh ax\cosh bx}{\cosh x}\,dx=\frac{\pi\cos\left(\frac{a\pi}{2}\right)\cos\left(\frac{b\pi}{2}\right)}{\cos a\pi+\cos b\pi}$$

$$-1<(a\pm b)<1.$$

$$I=\frac{1}{2}\int_0^{\infty}\frac{(e^{ax}+e^{-ax})\,(e^{bx}+e^{-bx})}{e^{x}+e^{-x}}\,dx$$

$$=\frac{1}{2}\int_0^{\infty}\frac{e^{(a+b)x}+e^{-(a+b)x}}{e^{x}+e^{-x}}+\frac{e^{(a-b)x}+e^{-(a-b)x}}{e^{x}+e^{-x}}\,dx=\frac{1}{2}(I_1+I_2).$$

Put $x=\pi z$; $\therefore$ $dx=\pi\,dz$.

$$\therefore\quad I_1=\frac{\pi}{2}\int_0^{\infty}\frac{e^{(a+b)\pi z}+e^{-(a+b)\pi z}}{e^{\pi z}+e^{-\pi z}}\,dz=\frac{\pi}{2}\sec\frac{(a+b)\pi}{2}.$$

Similarly, $I_2=\frac{\pi}{2}\sec\frac{(a-b)\pi}{2}$;

$$\therefore \quad I=\frac{1}{2}[I_1+I_2]=\frac{\pi}{4}\left[\frac{\cos(a+b)\pi/2+\cos(a-b)\pi/2}{2\cos(a+b)\pi/2.\cos(a-b)\pi/2}\right]$$

$$=\frac{\pi\cos\frac{a\pi}{2}\cos\frac{b\pi}{2}}{\cos a\pi+\cos b\pi}.$$

Note. *Below we shall give certain questions which after applying the principle of differentiation or integration under the sign of integration reduce to the standard form already discussed before in § 7 P. 64.*

Ex. 14. *Evaluate the following integrals :—*

(i) $\int_0^\infty \frac{x^{a-1}\log x}{1-x}\,dx$, (ii) $\int_0^\infty \frac{x^{a-1}\log x}{1+x}\,dx$.

(Agra 1948)

The forms of the above integrals give us an idea to reduce them to the standard form

$$\left.\begin{aligned}\int_0^\infty \frac{z^{a-1}}{1-z}\,dz&=\pi\cot a\pi,\\ \int_0^\infty \frac{z^{a-1}}{1+z}\,dz&=\pi\operatorname{cosec} a\pi\end{aligned}\right\}\text{R 3 and 4 P. 64}$$

Also remember that

$$\frac{d}{da}(z^a)=z^a\log z\,;\quad \because\quad \frac{d}{dx}(a^x)=a^x\log a.$$

Differentiating both the standard forms w.r.t. a under the sign of integration, we get

$$\int_0^\infty \frac{z^{a-1}\log z}{1-z}\,dz=-\pi^2\operatorname{cosec}^2 a\pi$$

and $$\int_0^\infty \frac{z^{a-1}\log z}{1+z}\,dz=-\pi^2\operatorname{coses} a\pi\cot a\pi.$$

Above gives us the values of the given integrals if we change the variable of integration to x instead of z.

Ex. 15. *prove that* $\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\,x\,dx=\frac{1}{4}\sec^2\frac{a}{2}$.

It follows at once by differentiating w r.t. a the result

$$\int_0^\infty \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}}\,dx=\frac{1}{2}\tan\frac{a}{2}.$$

Ex. 16. *Prove that* $\displaystyle\int_0^\infty \frac{x^r \log x}{1+x^2}\,dx=\frac{\pi^2}{4}\,\frac{\sin\frac{\pi r}{2}}{\cos^2\frac{\pi r}{2}}.$

(Agra 52, 56, 64 ; Rajputana 57)

We know that $\displaystyle\int_0^\infty \frac{x^r}{1+x^2}\,dx=\frac{\pi}{2}\sec\frac{r\pi}{2}.$ **[§ 8 P. 65]**

Hence if I be the given integral,

$$I=\int_0^\infty \frac{x^r \log x}{1+x^2}\,dx.$$

Integrating w.r.t r, we get

$$\int I\,dr=\int_0^\infty \frac{x^r}{1+x^2}\,dx$$

or

$$\int I\,dr=\frac{\pi}{2}\sec\frac{\pi r}{2}.$$

Differentiating both sides w.r.t. r, we get

$$I=\frac{\pi}{2}\cdot\frac{\pi}{2}\sec\frac{\pi r}{2}\tan\frac{\pi r}{2}=\frac{\pi^2}{4}\,\frac{\sin \pi r/2}{\cos^2 \pi r/2}.$$

Ex. 17. *Prove that* $\displaystyle\int_0^1 \frac{x^{a-1}-x^{-a}}{1+x}\,\frac{dx}{\log x}=\log\frac{\tan \pi a}{2}.$

(Rajputana 50)

We know that $\log\tan x/2$ is the integral of cosec x. Hence we should choose a formula, which involves cosec $a\pi$ whose integral is $\dfrac{2}{\pi}\log\tan\dfrac{\pi a}{2}$.

We know that $\displaystyle\int_0^\infty \frac{z^{a-1}}{1+z}\,dz=\pi\,\text{cosec}\,a\pi.$

Let the given integral be I and in order to eliminate $\log x$, we differentiate it w.r.t. a and remember that

$$\frac{d}{dx}(a^x)=a^x\log a\;;\quad\therefore\quad \frac{d}{da}(x^a)=x^a\log x.$$

$$\therefore \frac{dI}{da}=\int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}\log x \cdot \frac{dx}{\log x}=\int_0^1 \frac{x^{a-1}}{1+x}\,dx+\int_0^1 \frac{x^{-a}}{1+x}\,dx. \quad \ldots(1)$$

Now $\pi \operatorname{cosec} a\pi=\int_0^\infty \frac{x^{a-1}}{1+x}\,dx=\int_0^1 \frac{x^{a-1}}{1+x}\,dx+\int_1^\infty \frac{x^{a-1}}{1+x}\,dx.$

Put $x=\frac{1}{z}$ in the last integral and adjust the limits.

$$\therefore \quad \pi \operatorname{cosec} a\pi=\int_0^1 \frac{x^{a-1}}{1+x}\,dx+\int_1^0 \frac{\frac{1}{z^{a-1}}}{1+\frac{1}{z}}\left(-\frac{1}{z^2}\,dz\right)$$

$$=\int_0^1 \frac{x^{a-1}}{1+x}\,dx+\int_0^1 \frac{z^{-a}}{1+z}\,dz.$$

$$\pi \operatorname{cosec} a\pi=\int_0^1 \frac{x^{a-1}}{1+x}\,dx+\int_0^1 \frac{x^{-a}}{1+x}\,dz. \quad \ldots(2)$$

Hence from (1) and (2), we get

$$\frac{dI}{da}=\pi \operatorname{cosec} a\pi$$

Integrating both sides w.r.t. a, we get

$$I=\frac{\pi}{\pi}\log\tan\frac{\pi a}{2}+c=\log\tan\frac{\pi a}{2}+c.$$

When $a=\frac{1}{2}$, then clearly $I=0$ and $\log\tan\frac{\pi a}{2}$

$$=\log\tan\frac{\pi}{4}=\log 1=0\;;\quad \therefore\quad c=0.$$

$$\therefore \quad I=\log\tan\frac{\pi a}{2}.$$

Ex. 18. *Prove that* $\int_0^1 \frac{x^a+x^{-a}-2}{1-x}\frac{dx}{\log x}=\log\left(\frac{\sin a\pi}{a\pi}\right).$ (Agra 63)

If the given integral be I, then in order to eliminate $\log x$ as in Ex. 17.

$$\frac{dI}{da}=\int_0^1 \frac{x^a-x^{-a}}{1-x} \log x \frac{dx}{\log x}=\int_0^1 \frac{x^a-x^{-a}}{1-x}\, dx$$

or $$\frac{dI}{da}=\pi \cot a\pi-\frac{1}{a}$$ [See Ex. 5 P. 69 for proof]

Integrating both sides w.r.t. a, we get

$$I=\log \sin a\pi-\log a+c=\log \frac{\sin a\pi}{a}+c.$$

Now when $a=0$, then $I=0$

$$\therefore \quad 0=\log \left[\frac{\sin a\pi}{a\pi}\, \pi\right]+c.$$

Also we know that when $\theta \to 0$ $\frac{\sin \theta}{\theta}=1$.

$$\therefore \quad 0=\log \pi+c \quad \text{or} \quad c=-\log \pi.$$

$$\therefore \quad I=\log \frac{\sin (a\pi)}{a}-\log \pi=\log \left(\frac{\sin a\pi}{a\pi}\right).$$

CHAPTER II

BETA AND GAMMA FUNCTIONS

§ 1. Definitions. In B.Sc. integral calculus, we had stated that

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

We had also stated that $\Gamma(n+1) = n\,!$, $\Gamma\frac{1}{2} = \sqrt{\pi}$.

Here below we give the definitions of Beta and Gamma functions which define the first and second Eulerian integrals

as $\quad B(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1} \, dx \quad$ called Beta function

and $\quad \Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx \quad$ called Gamma function

and are read as Beta l, m and Gamma n respectively.

(Gauhati Hons. 63 ; Delhi Hons. 61, 55 ; Vikram 64)

The constants l, m, n occuring in the above functions are +ive numbers, which may be integrals or fractions.

§ 2. Properties.

(a) The function $B(l, m)$ is symmetrical w.r.t. l, m, i.e. $\quad B(l, m) = B(m, l)$.

$$B(l, m) = \int_0^1 x^{l-1} (1-x)^{m-1} \, dx.$$

Now by property IV of definite integrals,

$$\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx.$$

$$\therefore \quad B(l, m) = \int_0^1 (1-)^{l-1}[1-(1-x)]^{m-1}\, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{l-1}\, dx$$

or $$B(l, m) = B(m, l)$$ **Proved.**

(b) Evaluation of Beta function B (l, m).

$$B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx.$$

Let us suppose that **m is a +ive integer ;** then integrating by parts keeping $(1-x)^{m-1}$ as first function, we have

$$B(l, m) = \left[\frac{x^l}{l}(1-x)^{m-1}\right]_0^1 + \frac{(m-1)}{l}\int_0^1 x^l.(1-x)^{m-2}\, dx\ .$$

or $$B(l, m) = \frac{(m-1)}{l}\int_0^1 x^l(1-x)^{m-2}\, dx$$

Again integrating by parts as above, we get

$$B(l, m) = \frac{(m-1)}{l}\cdot\frac{(m-2)}{l+1}\int_0^1 x^{l+1}(1-x)^{m-3}\, dx.$$

Continuing the above process of integrating by parts, we get

$$B(l, m) = \frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots(l+m-2)}\int_0^1 x^{l+m-2}\, dx,$$

$$\mathbf{B(l, m)} = \frac{(m-1)\,!}{l(l+1)\ldots(l+m-2)}\cdot\frac{1}{(l+m-1)},$$ **m is a +ive integer.**

In case l alone is +ive integer, then since $B(l, m) = B(m, l)$, we can say that

$$\mathbf{B(l, m)} = \frac{(l-1)\,!}{m(m+1)\ldots(m+l-1)},$$ **l is a +ive integer.**

In case **both l and m are +ive integers,** then multiplying above and below by $1.2.3\ldots(l-1)$ or $1.2.3\ldots(m-1)$, we can say that

$$\mathbf{B(l, m)} = \frac{(m-1)\,!\,(l-1)\,!}{(l+m-1)\,!},$$ **both l and m are +ive integers**

(c) Evaluation of Gamma function n > 1.

To prove that $\Gamma n = (n-1)\,\Gamma(n-1)$.

(Karnatak 64 ; Nagpur 63 ; Delhi Hons. 56)

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x}\, dx.$$

Integrating by parts keeping x^{n-1} as first function,

$$\Gamma n = \left[-e^{-x} x^{n-1} \right]_0^\infty + (n-1) \int_0^\infty x^{n-2} e^{-x}\, dx.$$

Now $$\underset{x \to 0}{\text{Lt}}\ \frac{x^{n-1}}{e^x} = 0$$

and $$\underset{x \to \infty}{\text{Lt}}\ \frac{x^{n-1}}{e^x} = \underset{x \to \infty}{\text{Lt}}\ \frac{x^{n-1}}{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots}$$

$$= \underset{x \to \infty}{\text{Lt}}\ \frac{1}{\frac{1}{x^{n-1}} + \frac{1}{x^{n-2}.1!} + \frac{1}{x^{n-3}\,2!} + \ldots + \frac{x}{n!} + \ldots}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore\quad \Gamma n = (n-1) \int_0^\infty x^{n-2} e^{-x} = (n-1)\,\Gamma(n-1).$$

Hence we conclude that $\Gamma n = (n-1)\,\Gamma(n-1)$. **(Remember).**

Arguing as above, we can say that

$$\Gamma(n-1) = (n-2)\,\Gamma(n-2)$$

$$\therefore\quad \Gamma n = (n-1)(n-2)\,\Gamma(n-2).$$

Hence in case n be a +ive integer, then proceeding as above, we can say that

$$\Gamma n = (n-1)(n-2)(n-3)\ldots 3.2.1.\Gamma 1,$$

where $\Gamma 1 = \int_0^\infty x^{1-1} e^{-x}\, dx = \left[-e^{-x} \right]_0^\infty = -\left[\frac{1}{e^x} \right]_0^\infty = 1.$

$$\therefore\quad \Gamma n = (n-1)(n-2)(n-3)\ldots 3.2.1 = (n-1)\,!.$$

(Remember)

Hence we can say that

$\Gamma n = (n-1)\,\Gamma(n-1)$ for all values of n,

$\Gamma n = (n-1)\,!$ when n is a +ive integer,

and $\Gamma 1 = 1.$

Also it can be verified that $\Gamma 0=\infty$ and $\Gamma(-n)=\infty$, where n is a +ve integer.

Ex. *Prove that* $\int_0^\infty x^n e^{-ax}\,dx=\frac{n!}{a^{n+1}},\ a>0.$

Put $ax=t$ and proceed as above

§ 3. Transformation of Gamma functions.

We know that $\Gamma n=\int_0^\infty x^{n-1}e^{x}\,dx$ ...(1)

Let us put $x=ky$; $\therefore\ dx=k\,dy$.

$$\therefore\ \Gamma n=\int_0^\infty k^{n-1}y^{n-1}e^{-ky}.k\,dy=k^n\int_0^\infty y^{n-1}e^{-ky}\,dy.$$

(a) $\therefore\ \int_0^\infty y^{n-1}e^{-ky}\,dy=\frac{\Gamma n}{k^n}.$ **(Very Important)**

Again put $x^n=y$ in (1) ; $\therefore\ nx^{n-1}\,dx=dy$ and $-x=-y^{1/n}$.

$$\therefore\ \Gamma n=\frac{1}{n}\int_0^\infty e^{-y^{1/n}}\,dy.$$

(b) $\therefore\ \int_0^\infty e^{-y^{1/n}}\,dy=n\Gamma n=\Gamma(n+1).$

Cor. If we put $n=\frac{1}{2}$ in the above result, we get

$$\int_0^\infty e^{-y^2}\,dy=\tfrac{1}{2}\Gamma\tfrac{1}{2}.$$

But $$\int_0^\infty e^{-y^2}\,dy=\tfrac{1}{2}\sqrt{\pi}.$$ **(Ex. 3 P. 46)**

$$\therefore\ \frac{\sqrt{\pi}}{2}=\tfrac{1}{2}\Gamma\tfrac{1}{2}\ ;\quad \therefore\ \Gamma\tfrac{1}{2}=\sqrt{\pi}.$$ **(Remember)**

Again put $e^{-x}=y$ in (1) ; $\therefore\ -e^{-x}\,dx=dy$ and $e^x=\frac{1}{y}$.

$\therefore\ x=\log\frac{1}{y}$. When $x=0$, $y=1$ and when $x=\infty$, $y=0$.

$$\therefore\ \Gamma n=\int_1^0\left(\log\frac{1}{y}\right)^{n-1}(-dy)$$

or $$\Gamma n=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy.$$ **(Bombay 59)**

Ex. 1. *Evaluate* $\int_0^\infty e^{-x^2}dx$ *and hence show that* $\Gamma\frac{1}{2}=\sqrt{\pi}$.

See Ex 3 P 46 for first part and result (c) cor. above for 2nd part

§ 4. Transformation of Beta function.

We know that $B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx = B(m, l)$.

Put $x = \frac{1}{1+y}$; $\therefore\ dx = -\frac{1}{(1+y)^2}\,dy$.

$$1-x = 1-\frac{1}{1+y} = \frac{y}{1+y}$$

When $x=1$, $y=0$, when $x=0$, $y=\infty$.

$$\therefore\ B(l, m) = \int_\infty^0 \frac{1}{(1+y)^{l-1}}\cdot\frac{y^{m-1}}{(1+y)^{m-1}}\cdot\frac{-1}{(1+y)^2}\,dy$$

or $$B(l, m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}\,dy.$$

Again we know that $B(l, m)$ is symmetrical w.r.t. l, m;

hence $$B(l,m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}\,dy.$$

The latter form could be directly obtained if we put $x = \frac{y}{1+y}$ in the original form.

§ 5. Relation between Beta and Gamma functions.

To prove that $B(l, m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}$.

(Jiwaji 66; Gauhati Hons. 63; Vikram 64, 62; Gujrat 52; Agra 49, 51, 54, 62; Sagar 62, 66; Rajputana 52; Delhi Hons. 54, 56, 57, 61; Vikram 62, 64; Bombay 59; Punjab 60; Karnatak 64, 62)

From § 3 Result (a), we know that $\int_0^\infty x^{n-1}e^{-zx}\,dx = \frac{\Gamma(n)}{z^n}$

or $$\Gamma(l)=\int_0^\infty z^l x^{l-1} e^{-zx}\, dx. \qquad \ldots(1)$$

Also $$\Gamma(m)=\int_0^\infty z^{m-1} e^{-z}\, dz$$

Multiplying both sides of (1) by $z^{m-1}e^{-z}$, we get

$$\Gamma(l)\, z^{m-1}e^{-z}=\int_0^\infty z^{l+m-1} e^{-z\,(x+1)} x^{l-1}\, dx.$$

Integrating both sides w.r.t. z within limits 0 to ∞, we get

$$\Gamma(l)\int_0^\infty z^{m-1}e^{-z}\, dz=\int_0^\infty \left[\int_0^\infty z^{(l+m)-1}e^{-z(x+1)}\, dz\right] x^{l-1}\, dx.$$

But $\int_0^\infty z^{(l+m)-1}e^{-z(x+1)}\, dx=\frac{\Gamma(l+m)}{(1+x)^{l+m}}$ by result (a) § 3,

i.e. $$\int_0^\infty y^{n-1}e^{-ky}\, dy=\frac{\Gamma(n)}{k^n}.$$

$$\therefore\ \Gamma(l).\Gamma(m)=\int_0^\infty \Gamma(l+m).\frac{x^{l-1}}{(1+x)^{l+m}}dx$$

$$=\Gamma(l+m)\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\, dx.$$

$\therefore\ \Gamma(l)\ \Gamma(m)=\Gamma(l+m)\ B(l, m)$ by § 4.

$$\therefore\ B(l,m)=\frac{\Gamma(l).\Gamma(m)}{\Gamma(l+m)}.$$ **Proved.**

Deduction. Putting $l+m=1$, $m=(1-l)$ and $\Gamma(1)=1$, we get

$$\frac{\Gamma(l)\Gamma(1-l)}{1}=B(l,m)=\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}=\int_0^\infty \frac{x^{l-1}}{1+x}\, dx,$$

$$\because\ l+m=1.$$

But from § 7 Result 3 P. 64, we know that

$$\int_0^\infty \frac{x^{l-1}}{1+x}\, dx=\frac{\pi}{\sin l\pi},\ l<1.$$

$$\therefore \quad \Gamma(l)\ \Gamma(1-l)=\frac{\pi}{\sin l\pi}$$

or
$$\Gamma n\ \Gamma(1-n)=\frac{\pi}{\sin n\pi}. \qquad \ldots(2)$$

(Sagar 63 ; Agra 50, 52, 59, 62 ; Gujrat 59)

Putting $n=\frac{1}{2}$ in the above and $\sin\frac{\pi}{2}=1$, we get

$$\Gamma(\tfrac{1}{2})\ \Gamma(\tfrac{1}{2})=\pi \quad \text{or} \quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}.$$

We have proved that $\Gamma(n)\ \Gamma(1-n)=\frac{\pi}{\sin n\pi}$.

Multiply both sides by n and put $n\Gamma(n)=\Gamma(n+1)$.

$$\therefore \quad \Gamma(1+n)\ \Gamma(1-n)=\frac{n\pi}{\sin n\pi}. \qquad \ldots(3)$$

Exercise

Ex. 1. *Prove that*

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\ d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \qquad \textbf{(Remember)}$$

(Gauhati Hons. 65 ; Karnatak 64, 63 ; Rajputana 55)

or $$\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\ d\theta=\frac{\Gamma(m)\ \Gamma(n)}{2\Gamma(m+n)}.$$

We know that $B(m,\ n)=\int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx$

$$=\frac{\Gamma(m)\ \Gamma(n)}{\Gamma(m+n)}. \qquad \ldots(1)$$

Put $x=\sin^2\theta$; $\therefore\ dx=2\sin\theta\cos\theta\ d\theta$.

$$\therefore \quad \int_0^1 x^{m-1}\ (1-x)^{n-1}\ dx$$

$$=\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta.2\sin\theta\cos\theta\ d\theta$$

or $$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

$$\therefore\quad \int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}=\frac{1}{2}B(m,n).$$

But $2m-1=p$, i.e., $m=\dfrac{p+1}{2}$

and $2n-1=q$ or $n=\dfrac{q+1}{2}$.

$$\therefore\quad \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

Ex. 2. (a) *Evaluate the integral*

$$\int_a^b (x-a)^m (b-x)^n\,dx,$$

where m and n are +ive integers. **(Rajputana 60)**

Put $$\mathbf{x=a\sin^2\theta+b\cos^2\theta}.$$

$$\therefore\quad dx=-2(b-a)\sin\theta\cos\theta\,d\theta,$$

$$x-a=b\cos^2\theta-a(1-\sin^2\theta)=(b-a)\cos^2\theta,$$

$$b-x=b(1-\cos^2\theta)-a\sin^2\theta=(b-a)\sin^2\theta.$$

When $x=b$, then $\sin^2\theta=0$, $\therefore\ \theta=0$, and when $x=a$, $\cos\theta=0$; $\therefore\ \theta=\pi/2$.

$$\therefore\quad I=-\int_{\pi/2}^0 (b-a)^m\cos^{2m}\theta\,(b-a)^n\sin^{2n}\theta \times\{-2(b-a)\sin\theta\cos\theta\,d\theta\}$$

or $$I=2(b-a)^{m+n+1}\int_0^{\pi/2}\sin^{2n+1}\theta\cos^{2m+1}\theta\,d\theta$$

or $$I=2(b-a)^{m+n+1}\frac{\Gamma\frac{2n+1+1}{2}\,\Gamma\frac{2m+1+1}{2}}{2\Gamma\frac{2n+1+2m+1+2}{2}}.$$

$$I=(b-a)^{m+n+1}\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$=(b-a)^{m+n+1}\,B\,(m+1,\,n+1).$$

(b) *Prove that*

$$\int_a^b (x-a)^{m-1}\,(b-x)^{n-1}\,dx=(b-a)^{m+n-1}\,B(m,\,n).$$

(Delhi Hons. 61 ; Karnatak 64 ; Agra 45, 49)

Proceeding exactly as above,

$$I=(b-a)^{m+n-1}\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}=(b-a)^{m+n-1}\,B(m,\,n).$$

In can be easily verified if we put $m-1$ and $n-1$ for m and n respectively in the result of part (a)

Ex. 3. *Prove that* $\int_0^{\pi/2}\frac{d\theta}{\sqrt{(\sin\theta)}}\times\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta=\pi.$

(Karnatak 64 ; Agra 53, 55)

$$I=\int_0^{\pi/2}(\sin\theta)^{-1/2}\cos^0\theta\,d\theta\times\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$$

$$=\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)}\times\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)}.$$

$$=\frac{\Gamma\frac{1}{4}.\Gamma\frac{1}{2}}{2\Gamma\frac{3}{4}}\cdot\frac{\Gamma\frac{3}{4}\;\Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}}=\pi.$$

$\because\;\Gamma\frac{1}{2}=\sqrt{\pi}$ and $\Gamma\frac{5}{4}=\Gamma(\frac{1}{4}+1)=\frac{1}{4}\Gamma\frac{1}{4}$ as $\Gamma(n+1)=n\Gamma n.$

Ex. 4. *Prove that* $\int_0^1\frac{x^2\,dx}{(1-x^4)^{1/2}}\times\int_0^1\frac{dx}{(1+x^4)^{1/2}}=\frac{\pi}{4\sqrt{2}}.$

(Rajputana 63)

$$I=I_1\times I_2.$$

Putting $x^2=\sin\theta$ in I_1, we get

$$I_1=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{(\sin\theta)}}\cdot d\theta=\frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$$

Put $ay=bx$ etc. ; $\therefore$ $B(m, n)=a^m b^n \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.$

$$\therefore \quad I=\frac{1}{a^m b^n} B(m, n)=\frac{1}{a^m b^n} \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.$$

(c) $\int_0^\pi \frac{\sin^{n-1} x\,dx}{(a+b\cos x)^n}=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}.B\left(\frac{n}{2}, \frac{n}{2}\right).$

$$I=\int_0^\pi \frac{(2\sin x/2\cos x/2)^{n-1}\,dx}{\left[(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}\right]^n}$$

$$=\frac{2^{n-1}}{(a+b)^n}\int_0^\pi \frac{\tan^{n-1}\frac{x}{2}\sec^2\frac{x}{2}\,dx}{\left(1+\frac{a-b}{a+b}\tan^2\frac{x}{2}\right)^n} \quad \text{Put } \frac{a-b}{a+b}\tan^2\frac{x}{2}=t.$$

$$\therefore \quad I=\frac{2^{n-1}}{(a+b)^n}\int_0^\infty \frac{\left[\frac{a+b}{a-b}.t\right]^{(n-2)/2}}{(1+t)^n}.\frac{a+b}{a-b}\,dt$$

$$=\frac{2^{n-1}}{[(a+b)(a-b)]^{n/2}}\int_0^\infty \frac{t^{(n/2)-1}}{(1+t)^{\frac{n}{2}+\frac{n}{2}}}\,dt$$

$$=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}.B\left(\frac{n}{2}, \frac{n}{2}\right) \text{ by § 4 P. 82.}$$

(d) *Prove that* $\int_0^\pi \frac{\sqrt{(\sin x)}}{(5+3\cos x)^{3/2}}\,dx=\frac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{(2\pi)}}.$

Ex. 6. *Prove that* $\Gamma n.\Gamma(n+\frac{1}{2})=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n).$

(Gauhati Hons. 63 ; Delhi Hons. 58, 55, 53 ; Raj. 57 ; Indore 66 ; Agra, 46, 51, 55)

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{\Gamma m\,\Gamma n}{2\Gamma(m+n)}. \quad \ldots(1)$$

(Ex. 1 P. 84)

Put $n=\frac{1}{2}$ or $2n-1=0$.

$$\therefore \quad \int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta=\frac{\Gamma\frac{1}{2}\,\Gamma m}{2\Gamma(m+\frac{1}{2})}. \quad \ldots(2)$$

Again putting $m=n$ to get $\Gamma(2m)$ from (1),

$$\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta=\frac{(\Gamma m)^2}{2\Gamma(2m)}$$

or

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta=\frac{(\Gamma m)^2}{2\Gamma(2m)}.$$

Put $2\theta=t$ and adjust the limits.

$$\therefore\quad \frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1} t\,dt=\frac{(\Gamma m)^2}{2\Gamma(2m)}.$$

But $\int_0^{\pi} \sin^n x\,dx=2\int_0^{\pi/2} \sin^n x\,dx$ by Prop. VI.

$$\therefore\quad \frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} t\,dt=\frac{(\Gamma m)^2}{2\Gamma(2m)}$$

or

$$\frac{1}{2^{2m-1}}\,\frac{\Gamma\frac{1}{2}\,\Gamma m}{2\Gamma(m+\frac{1}{2})}=\frac{(\Gamma m)^2}{2\Gamma(2m)}$$

$$\therefore\quad \Gamma m\,\Gamma(m+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m).$$

You may replace m by n or we should have written gamma function as

$$\int_0^{\pi/2} \sin^{2n-1}\theta\cos^{2m-1}\theta\,d\theta \text{ and put } 2m-1=0.$$

Note. If we put $2n$ or $2m=p$, then the above result takes the form $\quad \Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right)=\frac{\sqrt{\pi}}{2^{p-1}}\,\Gamma p.$

Ex. 7. *Find the value of* $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$. **(Agra 52, 57)**

Put $\quad x^n=\sin^2\theta \quad$ or $\quad x=\sin^{2/n}\theta.$

$$\therefore\quad dx=\frac{2}{n}\sin^{(2/n)-1}\theta\cos\theta\,d\theta.$$

$$\therefore\quad I=\frac{2}{n}\int_0^{\pi/2}\sin^{(2/n)-1}\theta\,d\theta=\frac{2}{n}\,\frac{\Gamma\left[\frac{1}{2}\left(\frac{2}{n}-1+1\right)\right]\Gamma[\frac{1}{2}(0+1)]}{2\Gamma\left[\frac{1}{2}\left(\frac{2}{n}-1+0+2\right]\right.}$$

$$=\frac{2}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\frac{1}{2}}{2\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{n}\,\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.$$

Ex. 8. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$.

Putting $x^n=\sin^2\theta$ as in Ex. 7,

$$I=\frac{2}{n}\int_0^{\pi/2}\frac{\sin^{(2/n)-1}\theta\cos\theta\,d\theta}{\cos^{2/n}\theta}$$

$$=\frac{2}{n}\int_0^{\pi/2}\sin^{(2/n)-1}\theta\cos^{1-(2/n)}\theta\,d\theta.$$

$$I=\frac{2}{n}\,\frac{\Gamma\frac{1}{n}\,\Gamma\left(1-\frac{1}{n}\right)}{2\Gamma 1}$$

Now we know $\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$ and $\Gamma 1=1$. (P. 83)

$$\therefore\quad I=\frac{1}{n}\cdot\frac{\pi}{\sin \pi/n}.$$

If $I=\int_0^1\frac{dx}{(1-x^6)^{1/6}}$, then putting $n=6$, we get

$$I=\frac{1}{6}\,\frac{\pi}{\sin\pi/6}=\frac{\pi}{3}.$$

Ex. 9. *Prove that* $\int_0^1 (1-x^n)^{1/n}\,dx=\frac{1}{2n}\,\frac{\left[\Gamma\frac{1}{n}\right]^2}{2\Gamma\left(\frac{2}{n}\right)}$

Putting $x^n=\sin^2\theta$ as in Ex. 7,

$$I=\frac{2}{n}\int_0^{\pi/2}\sin^{(2/n)-1}\theta\cos^{1+(2/n)}\theta\,d\theta=\frac{2}{n}\,\frac{\Gamma\frac{1}{n}\,\Gamma\left(1+\frac{1}{n}\right)}{2\Gamma\left(\frac{2}{n}+1\right)}.$$

$$=\frac{2}{n}\cdot\frac{\Gamma\frac{1}{n}\cdot\left(\frac{1}{n}\right)\Gamma\frac{1}{n}}{2\cdot\frac{2}{n}\Gamma\frac{2}{n}}=\frac{1}{2n}\frac{\left[\Gamma\frac{1}{n}\right]^2}{\Gamma\frac{2}{n}}$$

Ex. 10. *Show that* $\int_0^{\pi/2} \tan^p \theta \, d\theta=\frac{\pi}{2}\sec\frac{p\pi}{2}$. **(Agra 49)**

$$I=\int_0^{\pi/2} \sin^p \theta \cos^{-p} \theta \, d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{2\Gamma\left(\frac{p-p+2}{2}\right)}.$$

Both $p+1$ and $1-p$ should be +ive i.e. >0.

In other words, $p>-1$ and $1>p$ i.e. $1>p>-1$.

$$\therefore \quad I=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(1-\frac{p+1}{2}\right)}{2\Gamma 1}.$$

It is of the form $\Gamma n\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}$ (P. 83)

$$\therefore \quad I=\frac{1}{2}\frac{\pi}{\sin\frac{p+1}{2}\pi}=\frac{\pi}{2}\frac{1}{\sin\left(\frac{\pi}{2}+\frac{p\pi}{2}\right)}$$

$$=\frac{\pi}{2}\cdot\frac{1}{\cos\frac{p\pi}{2}}=\frac{\pi}{2}\sec\frac{p\pi}{2}.$$

Ex. 11. *Prove that* $\Gamma(\frac{1}{2}-x)\,\Gamma(\frac{1}{2}+x)=(\frac{1}{4}-x^2)\,\pi\sec\pi x$.

We know that $\Gamma(n+1)=n\Gamma n$.

$$\therefore \quad \text{L.H.S.}=(\tfrac{1}{2}-x)\,\Gamma(\tfrac{1}{2}-x)\,(\tfrac{1}{2}+x)\,\Gamma(\tfrac{1}{2}+x)$$

$$=(\tfrac{1}{4}-x^2)\cdot\Gamma(\tfrac{1}{2}-x)\cdot\Gamma[1-(\tfrac{1}{2}-x)]$$

$$=(\tfrac{1}{4}-x^2)\cdot\frac{\pi}{\sin(\frac{1}{2}-x)\pi}=(\tfrac{1}{4}-x^2)\,\pi\sec\pi x.$$

$$\because \quad \Gamma n\Gamma(1-n)=\frac{\pi}{\sin n\pi}. \qquad \text{[R. 2 P. 83]}$$

Ex. 12. *Evaluate* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$

(Burdwan Hons. 64 ; Agra 57, 62)

We know that $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.$

$$I=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx=I_1+I_2 \text{ say.}$$

Put $x=\frac{1}{y}$ in I_2 ; $\therefore$ $I_2=\int_\infty^1 \frac{\left(\frac{1}{y}\right)^{n-1}}{\left(\frac{1+y}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy$

or $$I_2=\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy=\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx.$$

$$\therefore \quad I=\int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}}+\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

or $$I=B(m, n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.$$

Ex. 13. $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx,\ m>0, n>0.$ **(Agra 48**

$I=I_1-I_2$ and I_2 can be shown to be equal to I_1 as in Q. 12.

Otherwise also, $I_1=B(m, n)$ and $I_2=B(n, m)$; but we know that $B(m, n)=B(n, m)$; $\therefore$ $I_1=I_2$ or $I=0$.

Ex. 14. *Prove that* $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx=0.$

$$I=\int_0^\infty \frac{x^8}{(1+x)^{24}}-\int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$=\int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx-\int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

or $I=B(9, 15)-B(15, 9)=0$ $\quad\because\ B(m, n)=B(n, m)$

Ex. 15. *Prove that* $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}=\frac{1}{5005}$.

Proceeding as above, $I=B(5, 10)+B(10, 5)=2B(5, 10)$.

$$2.\frac{\Gamma 5.\Gamma 10}{\Gamma(5+10)}=2.\frac{4!\,9!}{14!}=2\,\frac{4.3.2.1.9!}{14.13.12.11.10.9!}=\frac{1}{5005}.$$

Ex. 16. *Evaluate* $\int_0^1 x^m(1-x^n)^p\,dx.$ **(Nagpur 62)**

We know that $\int_0^1 x^{l-1}(1-x)^{m-1}\,dx=B(l, m)=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$.

Put $x^n=z$ or $x=z^{1/n}$, $\therefore\ dx=\frac{1}{n}z^{(1/n)-1}\,dz$

and $x^m=z^{m/n}$.

$$\therefore\ I=\int_0^1 z^{m/n}(1-z)^p.\frac{1}{n}z^{(1/n)-1}\,dz$$

$$=\frac{1}{n}\int_0^1 z^{\frac{m+1}{n}-1}(1-z)^{(p+1)-1}.dz=\frac{1}{n}B\left(\frac{m+1}{n}, p+1\right)$$

$$=\frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}.$$

Ex. 17. (a) *Prove that*

$$\int_0^p x^m(p^q-x^q)^n\,dx=\frac{p^{qn+m+1}}{q}B\left(n+1,\frac{m+1}{q}\right)$$

if $p>0,\ q>0,\ m+1>0,\ n+1>0.$ **(Bombay 59)**

Put $x^q=p^q.z$ when $x=p$, $z=1$, $x=0$, $z=0$ etc. as in Q. 16.

Ex. 18. *Prove that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx=\frac{B(m, n)}{a^n(1+a)^m}$.

(Gauhati Hons. 63 ; Agra 58 ; Rajputana 53, 50)

$$B(m, n)=\int_0^1 y^{m-1}(1-y)^{n-1}\,dy. \qquad \ldots(1)$$

Put $y=\frac{(1+a)x}{a+x}=(1+a)\left(1-\frac{a}{a+x}\right)$; $\therefore\ dy=\frac{a(1+a)}{(a+x)^2}$.

When $x=1$, $y=1$, and when $x=0$, $y=0$.

Also $1-y=1-\frac{x(1+a)}{a+x}=\frac{a(1-x)}{a+x}$.

$$\therefore\quad B(m,n)=\int_0^1 \frac{(1+a)^{m-1}x^{m-1}}{(a+x)^{m-1}}\cdot\frac{a^{n-1}(1-x)^{n-1}}{(a+x)^{n-1}}\cdot\frac{a(1+a)}{(a+x)^2}\,dx$$

$$=(1+a)^m a^n\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx.$$

$$\therefore\quad I=\frac{1}{(1+a)^m a^n}B(m,n)=\frac{1}{(1+a)^m a^n}\frac{\Gamma m\Gamma n}{\Gamma(m+n)}.$$

(b) *Prove that* $\int_0^a \frac{x^{l-1}(a-x)^{m-1}\,dx}{(a+bx)^{l+m}}=\frac{B(l,m)}{a(1+b)^l}$.

(Sagar 64)

Put $y=\frac{(1+b)x}{a+bx}$ in $\int_0^1 y^{l-1}(1-y)^{m-1}\,dy=B(l,m)$.

Ex. 19. *Prove that*

$$2^n\,\Gamma(n+\tfrac{1}{2})=1.3.5\ldots(2n-1).\sqrt{\pi}.$$

We know that $\Gamma(n+1)=n\Gamma n$.

$$\therefore\quad \Gamma(n+\tfrac{1}{2})=\Gamma\left(\frac{2n+1}{2}\right)=\Gamma\left(\frac{2n-1}{2}+1\right)$$

$$=\frac{2n-1}{2}\,\Gamma\left(\frac{2n-1}{2}\right)=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\,\Gamma\left(\frac{2n-3}{2}\right)$$

$$=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

$\therefore\quad 2^n\,\Gamma(n+\frac{1}{2})=(2n-1).(2n-3).5.3.1\,\sqrt{\pi}.$

$\because\quad \Gamma\frac{1}{2}=\sqrt{\pi}$ and $1, 3, 5, \ldots (2n-1)$ are n numbers and hence the factor 2^n.

Ex. 20. *Prove that*

$$\int_0^\infty e^{-x^2}.x^{2n}\,dx=\sqrt{\pi}.\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\int_0^\infty e^{-x^2}.x^{2n+1}\,dx.$$

Put $x^2=z$ or $2x\,dx=dz$.

$$I=\int_0^\infty e^{-x^2}x^{2n-1}x\,dx=\tfrac{1}{2}\int_0^\infty e^{-z}z^{(2n-1)/2}\,dz=\tfrac{1}{2}\int_0^\infty e^{-z}z^{\frac{2n+1}{2}-1}\,dz$$

$=\frac{1}{2}\,\Gamma\frac{2n+1}{2}=\frac{1}{2}.\frac{(2n-1)\,(2n-3)\ldots5\;3.1}{2^n}\sqrt{\pi}$ by Q. 19.

$$\text{R.H.S.}=\frac{\sqrt{\pi}}{2^n . n\,!}\,[1.3.5. \quad (2n-1)].\tfrac{1}{2}\int_0^\infty e^{-z}z^{n+1-1}\,dz.$$

The integral is clearly $\Gamma(n+1)=n\,!$.

$$\therefore \quad \text{R.H.S}=\frac{\sqrt{\pi}}{2^n\,n\,!}.1.3\;5. \;.(2n-1).\tfrac{1}{2}\,(n\,!)$$

$$=\tfrac{1}{2}.\frac{1.3.5 \quad (2n-1)}{2^n}\sqrt{\pi}=\text{L.H S.} \quad \textbf{Proved.}$$

Ex. 21. *Show that the sum of the series*

$$\frac{1}{n+1}+m\,\frac{1}{n+2}+\frac{m\,(m+1)}{2\,!}\,\frac{1}{n+3}+\frac{m\,(m+1)\,(m+2)}{3\,!}.\frac{1}{n+4}+\ldots$$

is $+\ldots$ *is* $\dfrac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)}$.

We know that

$$B(n+1,\,m+1)=\int_0^1 x^n\,(1-x)^m\,dx=\frac{\Gamma(n+1)\,\Gamma(m+1)}{\Gamma(m+n+2)}.$$

Putting $-m$ for m, we get

$$\frac{\Gamma(n+1).\Gamma(1-m)}{\Gamma(n-m+2)}=\int_0^1 x^n\,(1-x)^{-m}\,dx$$

$$=\int_0^1 x^n\left(1+mx+\frac{m\,(m+1)}{2\,!}\,x^2+\frac{m\,(m+1)\,(m+2)}{3\,!}\,x^3+\ldots\right)dx$$

$$=\left[\frac{x^{n+1}}{n+1}+m.\frac{x^{n+2}}{n+2}+\frac{m\,(m+1)}{2\,!}\,\frac{x^{n+3}}{n+3}+\frac{m\,(m+1)\,(m+2)}{3\,!}.\frac{x^{n+4}}{n+4}+\ldots\right]_0^1$$

$$=\frac{1}{n+1}+m.\frac{1}{n+2}+\frac{m\,(m+1)}{2\,!}.\frac{1}{n+3}+\frac{m\,(m+1)\,(m+2)}{3\,!}.\frac{1}{n+4}+\ldots$$

Ex. 22. *By means of the integral* $\int_0^1 x^{m-1}\,(1-x^a)^n\,dx$, *prove that*

$$\frac{1}{m.n\,!}-\frac{1}{(m+a)\,\{(n-1)\,!\}.1\,!}+\frac{1}{(m+2a).\{(n-2)\,!\}.2\,!}\cdots$$

$$\cdots+\frac{(-1)^n}{(m+na)\,(n)\,!}=\frac{a^n}{m\,(m+a)\,(m+2a)\ldots(m+na)}.$$

$$I=\int_0^1 x^{m-1}\left[1-nx^a+\frac{n(n-1)}{2!}x^{2a}+\ldots\right]$$

$$=\left[\frac{x^m}{m}-n\frac{x^{m+a}}{m+a}+\frac{n(n-1)}{2!}\frac{x^{m+2a}}{m+2a}\ldots\right]_0^1.$$

$$I=\frac{1}{m}-n.\frac{1}{m+a}+\frac{n(n-1)}{2!}.\frac{1}{m+2a}\ldots$$

Again putting $x^a=z$ or $x=z^{1/a}$,

$$x^{m-1}=z^{(m-1)/a},\ dx=\frac{1}{a}z^{(1/a)-1}.$$

$$\therefore\quad I=\int_0^1 z^{(m-1)/a}.(1-z)^n\frac{1}{a}z^{(1/a)-1}dz$$

$$=\frac{1}{a}\int_0^1 z^{(m/a)-1}(1-z)^n dz$$

or
$$I=\frac{1}{a}B\left(\frac{m}{a},n+1\right)=\frac{1}{a}\frac{\Gamma\left(\frac{m}{a}\right).\Gamma(n+1)}{\Gamma\left(\frac{m}{a}+n+1\right)}$$

$$=\frac{1}{a}\frac{n!.\Gamma\left(\frac{m}{a}\right)}{\left(\frac{m}{a}+n\right)\left(\frac{m}{a}+n-1\right)\ldots\left(\frac{m}{a}+1\right)\left(\frac{m}{a}\right).\Gamma\frac{m}{a}}$$

or
$$I=\frac{a^{n+1}}{a}\frac{n!}{(m+na)\ldots(m+2a)(m+a)m}$$

$$=\frac{a^n n!}{(m+na)\ldots(m+2a)(m+a)m}.$$

$\therefore$ Equating the values of I, we get

$$\frac{1}{m}-n\frac{1}{m+a}+\frac{n(n-1)}{2!}\frac{1}{m+2a}-\ldots$$

$$=\frac{a^n.n!}{(m+na)\ldots(m+2a)(m+a)m}.$$

Also from author's Trigonometery for B.Sc. Classes, [Deduction (b) P], we know that

$$\frac{\sin n\theta}{\sin \theta}=2^{n-1} \sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots$$

$$\ldots \sin\left(\theta+\frac{n-2}{n}\pi\right)\sin\left(\theta+\frac{n-1}{n}\pi\right).$$

Putting $\theta=0$, $\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin n\theta}{\sin\theta}=\underset{\theta\to 0}{\text{Lt}}\ \frac{n\cos n\theta}{\cos\theta}=n.$

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ \ldots\ \sin\frac{n-2}{n}\pi\sin\frac{n-1}{n}\pi. \quad \ldots(4)$$

$$\therefore\quad P^2=\pi^{n-1}\ \frac{2^{n-1}}{n},\quad \therefore\quad P=\frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}.$$

Hence $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\ \ldots\ \Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}.$

If we put $n=9$, we get

$\Gamma\frac{1}{9}\,\Gamma\frac{2}{9}\ \ldots\ \Gamma\frac{8}{9}=\frac{(2\pi)^{(9-1)/2}}{\sqrt{9}}=\frac{(2\pi)^4}{3}=\frac{16}{3}\pi^4.$ **(Rajputana 50)**

If we put $n=10$, we get

$$\Gamma\tfrac{1}{10}\,\Gamma\tfrac{2}{10}\ldots\ \Gamma\tfrac{9}{10}=\frac{(2\pi)^{(10-1)/2}}{\sqrt{10}}$$

or $$\Gamma(\cdot 1)\,\Gamma(\cdot 2)\ldots\Gamma(\cdot 9)=\frac{(2\pi)^{9/2}}{\sqrt{10}}.$$

§ 7. Real and imaginary parts.

Evaluate the integrals :

$$\int_0^\infty e^{-ax}\cos bx\,.\,x^{m-1}\,dx \text{ and } \int_0^\infty e^{-ax}\sin bx\,.\,x^{m-1}\,dx.$$

(Agra 60)

Both the above integrals are respectively the real and imaginary parts of

$$\int_0^\infty e^{-ax}.e^{ibx}.x^{m-1}\,dx \quad\text{or}\quad \text{of} \int_0^\infty e^{-(a-ib)x}.x^{m-1}\,dx.$$

$$\because\quad \cos\theta+i\sin\theta=e^{i\theta}.$$

Now from § 3 (a) P 81, we know that

$$\int_0^\infty e^{-kx}\, x^{n-1}\, dx = \frac{\Gamma n}{k^n}$$

$$\therefore \quad \int_0^\infty e^{-(a-ib)x}\, x^{m-1}\, dx = \frac{\Gamma(m)}{(a-ib)^m} = \Gamma m \cdot \frac{(a+ib)^m}{(a^2+b^2)^m}.$$

Let us put $a = r\cos\theta$, $b = r\sin\theta$ in R.H.S.

$$\therefore \quad \int_0^\infty e^{-ax}\,(\cos bx + i\sin bx)\, x^{m-1}\, dx = \Gamma m\, \frac{r^m(\cos\theta + i\sin\theta)^m}{r^{2m}}$$

or

$$\int_0^\infty [e^{-ax}\cos bx \,.\, x^{m-1} + ie^{-ax}\sin bx \,.\, x^{m-1}]\, dx = \frac{\Gamma m}{r^m}(\cos m\theta + i\sin m\theta).$$

Equating real and imaginary parts, we get

$$\int_0^\infty (e^{-ax}\cos bx \,.\, x^{m-1}\, dx = \frac{\Gamma m}{r^m}\cos m\theta$$

and

$$\int_0^\infty e^{-ax}\sin bx \,.\, x^{m-1}\, dx = \frac{\Gamma m}{r^m}\sin m\theta$$

where $r = \sqrt{(a^2+b^2)}$ and $\theta = \tan^{-1}\dfrac{b}{a}$.

Exercise

Ex. 1. *Prove that*

$$\int_0^\infty xe^{-ax}\cos bx \,.\, dx = \frac{a^2-b^2}{(a^2+b^2)^2} \text{ where } a > 0$$

Put $m-1=1$, *i.e.*, $m=2$ in § 7 P. 98 ; we get

$$\int_0^\infty xe^{-ax}\cos bx\, dx = \frac{\Gamma 2}{r^2}\cos 2\theta = \frac{1}{(a^2+b^2)} \cdot \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1}{(a^2+b^2)} \cdot \frac{1-b^2/a^2}{(1+b^2/a^2)} = \frac{a^2-b^2}{(a^2+b^2)^2}.$$

Note. Students should proceed directly as in § 7 with

the calculation of

$$\int_0^\infty x^{2-1}e^{-ax}.e^{ibx}=\int_0^\infty e^{-(a-ib)x}\,x^{2-1}\,dx=\frac{\Gamma 2}{(a-ib)^2}$$

$$=1.\frac{(a+ib)^2}{(a^2+b^2)^2}=\frac{a^2-b^2+i.2ab}{(a^2+b^2)^2}.$$

Put $e^{ibx}=\cos bx+i\sin bx$ and equate real and imaginary parts.

$$\therefore\quad \int_0^\infty xe^{-ax}\cos bx\,dx=\frac{a^2-b^2}{(a^2+b^2)^2}$$

and

$$\int_0^\infty xe^{-ax}\sin bx\,dx=\frac{2ab}{(a^2+b^2)^2}.$$

Alternative Method.

We know $\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$ Q. 3 (c) P. 30.

$$\therefore\quad \frac{dI}{da}=\int_0^\infty -xe^{-ax}\cos bx\,dx=\frac{(a^2+b^2).1-a.2a}{(a^2+b^2)^2}$$

$$=-\frac{(a^2-b^2)}{(a^2+b^2)^2}.$$

$$\therefore\quad \int_0^\infty xe^{-ax}\cos bx=\frac{a^2-b^2}{(a^2+b^2)^2}.$$

Again $\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{(a^2+b^2)}$,

$$\therefore\quad \frac{dI}{da}=\int_0^\infty -xe^{-ax}\sin bx\,dx=-\frac{b.2a}{(a^2+b^2)^2}.$$

$$\therefore\quad \int_0^\infty xe^{-ax}\sin bx\,dx=\frac{2ab}{(a^2+b^2)^2}.$$

Ex. 2. *Evaluate*

$$\int_0^\infty x^{m-1}\cos bx\,dx \text{ and } \int_0^\infty x^{m-1}\sin bx\,dx.$$

We know that

$$\int_0^\infty e^{-(-ib)x}\,x^{m-1}\,dx=\frac{\Gamma m}{(-ib)^m}=\frac{\Gamma m}{b^m}\frac{(0+i)^m}{(0-i^2)^m}$$

$$=\frac{\Gamma m}{b^m}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^m=\frac{\Gamma m}{b^m}\cos\left(\frac{m\pi}{2}+i\sin\frac{m\pi}{2}\right).$$

Equating real and imaginary parts, we get the value of given integrals as

$$\frac{\Gamma m}{b^m} \cos \frac{m\pi}{2} \text{ and } \frac{\Gamma m}{b^m} \sin \frac{m\pi}{2} \text{ respectively.}$$

It follows at once from § 7 P 98 by putting $a=0$ so that

$$\tan \theta = \frac{b}{0} = \infty \quad \text{or} \quad \theta = \frac{\pi}{2} \text{ and } r = \sqrt{(0+b^2)} = b.$$

Ex. 3. $\int_0^\infty \cos (bz^{1/n})\, dz = \dfrac{\Gamma(n+1) \cos \frac{n\pi}{2}}{b^n}$. **(Vikram 64)**

Put $z^{1/n} = x$, $\therefore$ $z = x^n$ or $dz = nx^{n-1}$

$I = n \int_0^\infty x^{n-1} \cos bx\, dx = n . \dfrac{\Gamma n}{b^n} \cos \dfrac{n\pi}{2}$ by Q 2.

[Students should prove the above result as in Q. 2]

But $\qquad \Gamma(n+1) = n\Gamma n$

$$\therefore \quad I = \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2}. \qquad \textbf{Proved.}$$

Ex. 4. *Prove that* $\int_0^\infty x^{2n-1} e^{-ax^2}\, dx = \dfrac{\Gamma n}{2a^n}$.

Put $\quad x^2 = z$; $\therefore$ $2x\, dx = dz$.

$$\therefore \quad I = \int_0^\infty x^{(2n-2)}\, e^{-ax^2}\, x\, dx = \frac{1}{2} \int_0^\infty z^{n-1} e^{-az}\, dz$$

$$= \frac{1}{2} . \frac{\Gamma n}{a^n}. \qquad \textbf{Proved.}$$

Ex. 5. Evaluate $\int_0^\infty \dfrac{\cos bz}{z^n}\, dz$ *and* $\int_0^\infty \dfrac{\sin bz}{z^n}\, dz$.

We know the following formulae :

$$\int_0^\infty e^{-az} x^{n-1}\, dx = \frac{\Gamma n}{a^n},$$

$$\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2} \text{ and } \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}.$$

Now $\qquad \dfrac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} . x^{n-1}\, dx. \qquad \ldots(1)$

$$\therefore \quad \frac{\Gamma n}{z^n} \cos bz = \int_0^\infty e^{-zx} . \cos bz . x^{n-1} \, dx.$$

Integrating both sides w.r.t. z within limits 0 to ∞,

$$\Gamma n \int_0^\infty \frac{\cos bz}{z^n} \, dz = \int_0^\infty \left[\int_0^\infty e^{-zx} \cos bz \, dz \right] x^{n-1} \, dx.$$

$$= \int_0^\infty \frac{x}{b^2 + x^2} . x^{n-1} \, dx = \int_0^\infty \frac{x^n}{b^2 + x^2} \, dx.$$

Again put $x = by$.

$$\therefore \quad \Gamma n \; I = \int_0^\infty \frac{b^n \; y^n}{b^2 \, (1 + y^2)} \, b \, dy = \int_0^\infty b^{n-1} \frac{y^n}{1 + y^2} \, dy.$$

$$\therefore \quad I = \frac{b^{n-1}}{\Gamma n} \frac{\pi}{2} \sec \frac{n\pi}{2}, \quad \because \int_0^\infty \frac{x^n}{1 + x^2} \, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}.$$

[§ 8 P. 65]

Again multiply both sides of (1) by $\sin bz$.

$$\therefore \quad \frac{\Gamma n}{z^n} \sin bz = \int_0^\infty e^{-zx} \sin bz \, x^{n-1} \, dx.$$

Integrating both sides w.r.t. z within limits 0 to ∞,

$$\Gamma n \int_0^\infty \frac{\sin bz}{z^n} \, dz = \int_0^\infty \left[\int_0^\infty e^{-zx} \sin bz \, dz \right] x^{n-1} \, dx.$$

$$\therefore \quad \Gamma n \; I = \int_0^\infty \frac{b}{b^2 + x^2} \, x^{n-1} \, dx.$$

Again put $x = by$

$$\therefore \quad \Gamma n . I = \int_0^\infty \frac{b^n . y^{n-1}}{b^2 \, (1 + y^2)} \, (b \, dy) = b^{n-1} \int_0^\infty \frac{y^{n-1}}{1 + y^2} \, dy$$

or

$$I = \frac{b^{n-1}}{\Gamma n} . \frac{\pi}{2} \sec (n-1) \frac{\pi}{2} \qquad [\S \, 8 \text{ P. } 65]$$

$$= \frac{b^{n-1}}{\Gamma n} . \frac{\pi}{2} \operatorname{cosec} \frac{n\pi}{2}.$$

$$\therefore \quad \sec (n-1) \frac{\pi}{2} \sec - \left(\frac{n\pi}{2} - \frac{\pi}{2} \right) = \operatorname{cosec} \frac{n\pi}{2}.$$

CHAPTER III

DIRICHLET'S THEOREM

§ 1. Double Integration. In cartesian co-ordinates the double integration $\int_a^b \int_0^{f(x)} dx\, dy$ represents the area of the curve $y=f(x)$ enclosed between ordinates $x=a$ and $x=b$. In the above notation the right hand element indicates the first integration.

Similarly in polar co-ordinates the double integration $\int_\alpha^\beta \int_0^{f(\theta)} r\, d\theta\, dr$ represents the area of the curve $r=f(\theta)$ enclosed between two radii vectors drawn at the points whose vectorial angles are α and β respectively. The right hand element denotes the first integration.

Similarly under suitable limits the triple integral $\iiint dx\, dy\, dz$ stands for the volume of a solid.

§ 2. Dirichlet's Theorem. The theorem states that

$$\int \cdots \int x_1^{l_1-1} \cdot x_2^{l_2-1} \cdots x_n^{l_n-1} dx_1\, dx_2 \ldots dx_n$$

$$=\frac{\Gamma(l_1)\,\Gamma(l_2)\ldots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\ldots l_n)}, \qquad \textbf{(Remember)}$$

where the integral is extended to all positive values of the variables subject to the condition $x_1+x_2+\ldots x_n \leqslant 1$.

(Calcutta Hons. 61 ; Vikram 65, 64 ; Agra 53, 50 ; Punjab 52 ; Nagpur 62)

We shall prove the above theorem by the method of induction, *i. e* first prove it for $n=2$, 3 and then assuming it to hold for $n=n$, we shall show that it also holds good for $n=n+1$ and hence universally true

Two Variables.

$$\iint x_1^{l_1-1}\, x_2^{l_2-1}\, dx_1\, dx_2 \qquad (\text{where } x_1+x_2 \leqslant 1)$$

$$=\frac{\Gamma(l_1).\Gamma(l_2)}{\Gamma(1+l_1+l_2)}. \qquad \text{(Karnatak 62)}$$

Let us denote the above double integral by I_2.

$$\therefore \quad I_2=\int_0^1 \int_0^{1-x_1} x_1^{l_1-1}.x_2^{l_2-1}\, dx_1\, dx_2 \qquad (x_1+x_2 \leqslant 1)$$

$$=\int_0^1 x_1^{l_1-1}\left[\frac{(x_2^{l_2})}{l_2}\right]_0^{1-x_1} dx_1$$

$$=\frac{1}{l_2}\int_0^1 x_1^{l_1-1}.(1-x_1)^{l_2+1-1}\, dx_1$$

$$=\frac{1}{l_2}\ B(l_1, l_2+1)=\frac{1}{l_2}\ \frac{\Gamma(l_1)\ \Gamma(l_2+1)}{\Gamma(1+l_1+l_2)}$$

$$=\frac{\Gamma(l_1).\Gamma(l_2)}{\Gamma(1+l_1+l_2)}. \qquad \ldots(1)$$

$$\because \quad \Gamma(n+1)=n\Gamma n.$$

Above shows that the theorem holds good for $n=2$.

Particular Case. In case $x_1+x_2 \leqslant h$ (instead of $x_1+x_2 \leqslant 1$), then put $\frac{x_1}{h}=u_1$ and $\frac{x_2}{h}=u_2$, so that $u_1+u_2 \leqslant 1$.

Also $dx_1=h\, du_1$ and $dx_2=h\, du_2$, so that

$$I_2=\iint x_1^{l_1-1}.x_2^{l_2-1}\, dx_1\, dx_2$$

$$=\iint (hu_1)^{l_1-1}.(hu_2)^{l_2-1}\,(h\, du_1)\,(h\, du_2)$$

$$=h^{l_1+l_2}\iint u_1^{l_1-1}.u_2^{l_2-1}\, du_1\, du_2 \qquad \text{where } u_1+u_2 \leqslant 1.$$

$$I_2=\frac{\Gamma(l_1).\Gamma(l_2)}{\Gamma(1+l_1+l_2)}.h^{l_1+l_2}, \text{ if } x_1+x_2 \leqslant h. \qquad \ldots(2)$$

Hence the changing of condition from $x_1+x_2 \leqslant 1$ to $x_1+x_2 \leqslant h$ gives rise to the factor $h^{l_1+l_2}$ in the answer.

$$-\int x_1^{l_1-1}\,dx_1 \iint\cdots\int x_2^{l_2-1}.x_3^{l_3-1}\cdots (x_{n+1})^{l_{n+1}-1}\,dx_2\ldots dx_{n+1}$$

$$=\int_0^1 x_1^{l_1-1}\,dx_1\,(1-x_1)^{l_2+l_3+\ \ l_{n+1}} \times\frac{\Gamma(l_2)\,\Gamma(l_3)\,..\Gamma(l_{n+1})}{\Gamma(1+l_2+l_3\ldots l_{n+1})}$$

$$=\frac{\Gamma(l_2).\Gamma(l_3)\ ..\Gamma(l_{n+1})}{\Gamma(1+l_2+l_3+\ldots+l_{n+1})}\times\frac{\Gamma(l_1).\Gamma(l_2+l_3+\ldots l_{n+1}+1)}{\Gamma(1+l_1+l_2+\ldots l_{n+1})}$$

$$=\frac{\Gamma(l_1).\Gamma(l_2)\ \ .\Gamma(l_{n+1})}{\Gamma(1+l_1+l_2+\ \ \ +l_{n+1})}.$$

Above shows that the theorem holds for $(n+1)$ variables as well Also we have actually shown that the theorem holds for two and three variables Hence the theorem holds good for any number of variables

§ 3. Liouville's extension of Dirichlet's Theorem.

Statement. If x, y, z are all +ive such that

$$h_1 < (x+y+z) < h_2,$$

then $$\iiint x^{l-1}y^{m-1}z^{n-1}\,F(x+y+z)\,dx.dy.dz =\frac{\Gamma l\ \Gamma m.\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} F(h)\,h^{l+m+n-1}\,dh.$$

(Rajputana 49, 55, 58 ; Agra 52, 57, 64)

We have by Dirichlet's Theorem that if $x+y+z < h$, then $$I=\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz=\frac{\Gamma l.\Gamma m.\Gamma n}{\Gamma(1+l+m+n)}.h^{l+m+n}. \quad \ldots(1)$$

Again if $x+y+z \leqslant h+\delta h$, then

$$I=\frac{\Gamma l.\Gamma m.\Gamma n}{\Gamma(1+l+m+n)}.(h+\delta h)^{l+m+n}. \quad \ldots(2)$$

From (1) and (2), we conclude that the value of I

extended to all such positive values of the variables such that their sum lies between h and $h+\delta h$ is

$$I=\frac{\Gamma l.\Gamma m.\Gamma n}{\Gamma(1+l+m+n)}\ [(h+\delta h)^{l+m+n}-h^{l+m+n}].$$

To a first order of approximation,

$$(x+\delta x)^n-x^n=nx^{n-1}\ \delta x$$

$$\therefore\quad I=\frac{\Gamma l\Gamma m\Gamma n}{\Gamma(1+l+m+n)}\ (l+m+n)\ h^{l+m+n-1}\ \delta h.$$

Now $\qquad\Gamma(1+n)=n\Gamma n$

$$\therefore\quad I=\frac{\Gamma l\Gamma m\Gamma n}{\Gamma(l+m+n)}\ h^{l+m+n-1}\ \delta h$$

Hence the integral

$$\iiint F(x+y+z)\ x^{l-1}y^{m-1}z^{n-1}\ dx\ dy\ dz$$

taken between the same limits has for its value

$$\frac{\Gamma l\Gamma m\Gamma n}{\Gamma(l+m+n)}\ F(h)\ h^{l+m+n-1}\ \delta h.$$

When $h_1<(x+y+z)<h_2$, the value of integral is

$$\frac{\Gamma l\Gamma m\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} F(h)\ h^{l+m+n-1}\ dh$$

Now we shall illustrate the above articles by giving suitable examples.

Exercise

Ex. 1. *Evaluate the integral* $\iiint x^{l-1}y^{m-1}z^{n-1}\ dx\ dy\ dz$, *where x, y, z are all $+$ive but limited by the condition*

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r\leqslant 1.$$

(Vikram 62; Sagar 63 ; Agra 50, 53, 56 ; Punjab 52)

In order to reduce the above integral to Dirichlet's form, we put $\left(\frac{x}{a}\right)^p=u$ or $x=au^{1/p}$; $\therefore\quad dx=\frac{a}{p}\ u^{(1/p)-1}\ du$ etc.

$\therefore \quad x^{l-1}\, dx = a^{l-1} u^{(l-1)/p} \cdot \frac{a}{p} \cdot u^{(1-p)/p}\, du = \frac{a^l}{p}\, u^{(l/p)-1}\, du.$

Similarly put $\left(\frac{y}{b}\right)^q = v$ and $\left(\frac{z}{c}\right)^r = w$, etc.

Hence subject to the condition that $u+v+w < 1$, the given integral is

$$I = \frac{a^l}{p} \frac{b^m}{q} \cdot \frac{c^n}{r} \iiint u^{(l/p)-1}\, v^{(m/q)-1}\, w^{(n/r)-1}\, du\, dv\, dw$$

$$= \frac{a^l \cdot b^m \cdot c^n}{p\ q\ r} \frac{\Gamma\left(\frac{l}{p}\right) \cdot \Gamma\left(\frac{m}{q}\right) \cdot \Gamma\left(\frac{n}{r}\right)}{\Gamma\left(1+\frac{l}{p}+\frac{m}{q}+\frac{n}{r}\right)} \text{ by Dirichlet's Theorem.}$$

Ex. 2. *Show that if l, m, n are all +ive,*

$$\iiint x^{l-1} \cdot y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{a^l b^m c^n}{8} \frac{\Gamma\left(\frac{l}{2}\right) \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{l+m+n+2}{2}\right)},$$

where the multiple integral is taken throughout that part of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies in the +ive octant.

(Punjab 60)

For points inside the positive octant of ellipsoid, we have
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1.$$

Hence proceeding as in Ex. 1 or putting $p=q=r=2$ in the result of Ex. 1, we get the required result at once.

Ex. 3. *Show that*

$$\iint x^{2l-1} y^{2m-1}\, dx\, dy = \tfrac{1}{4} c^{2(l+m)} \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l+m+1)}$$

for all +ive values of x and y lying inside the circle $x^2+y^2=c^2$.

As in Ex. 2, we have $x^2+y^2 \leqslant c^2$ or $\frac{x^2}{c^2}+\frac{y^2}{c^2} \leqslant 1$.

Put $\frac{x^2}{c^2}=u$ and $\frac{y^2}{c^2}=v$, where $u+v \leqslant 1$

Now proceeding as in Ex. 1, we get the required result.

Ex. 4. **Evaluate** $\iiint xyz\, dx\, dy\, dz$ *taking throughout the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$.

Put $\frac{x^2}{a^2}=u$, $\therefore$ $x=a\,(u)^{1/2}$ or $dx=\frac{a}{2}\, u^{-1/2}\, du$.

$$\therefore \quad x\, dx=\frac{a^2}{2}\, du=\frac{a^2}{2}\, u^{1-1}\, du.$$

Let us evaluate the given integral for +ive octant first.

$$\therefore \quad I=\iiint \frac{a^2}{2}\,\frac{b^2}{2}\,\frac{c^2}{2}\, u^{1-1}v^{1-1}w^{1-1}\, du\, dv\, dw,$$

where $u+v+w \leqslant 1$ for +ive octant

$$=\frac{a^2b^2c^2}{8}\;\frac{\Gamma 1\Gamma 1\Gamma 1}{\Gamma(1+1+1+1)}=\frac{a^2b^2c^2}{8\Gamma 4}=\frac{a^2b^2c^2}{8\,(3\,!)}=\frac{a^2b^2c^2}{48}.$$

Hence for the whole ellipsoid, the given integral is $8\times\frac{a^2b^2c^2}{48}=\frac{a^2b^2c^2}{6}$ as there are 8 octants.

Ex. 5. *Evaluate* $\iiint dx\, dy\, dz$, *where* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$.

Proceeding as above,

$$I=\iiint \frac{a}{2}\cdot\frac{b}{2}\cdot\frac{c}{2}\, u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\, du\, dv\, dw,$$

where $u+v+w < 1$

$$=\frac{abc}{8}\cdot\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}=\frac{abc}{8}\cdot\frac{(\sqrt{\pi})^3}{\Gamma(\frac{3}{2}+1)}=\frac{abc}{8}\cdot\frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}$$

$$=\pi\,\frac{abc}{6}.$$

Ex. 8. *Find the value of* $\iiint \log(x+y+z)\,dx\,dy\,dz$, *the integral extending over all +ive values of* x, y, z, *subject to the condition* $x+y+z<1$ **(Agra 45, 54, 57)**

Here x, y, z are all +ive and their sum is less than 1, *i.e.* $0<(x+y+z)<1$.

$$\therefore \quad I=\iiint \log(x+y+z)\,x^{1-1}y^{1-1}z^{1-1}\,dx\,dy\,dz$$

$$=\frac{\Gamma 1.\Gamma 1.\Gamma 1}{\Gamma(1+1+1)}\int_0^1 \log(h)\,h^{1+1+1-1}\,dh$$

$$=\frac{1}{\Gamma 3}\int_0^1 h^2\log h\,dh$$

$$=\frac{1}{2}\left[\frac{h^3}{3}\log h-\frac{1}{3}\int h^2\,dh\right]$$

$$=\frac{1}{2}\left[\frac{h^3}{3}\log h-\frac{h^3}{9}\right]_0^1=-\frac{1}{18}.$$

$$\underset{x\to 0}{\text{Lt}}\,x^3\log x=0\times\infty=\underset{x\to 0}{\text{Lt}}\frac{\log x}{1/x^3}=\underset{x\to 0}{\text{Lt}}\frac{1/x}{-3/x^4}$$

$$=\underset{x\to 0}{\text{Lt}}\frac{-x^3}{3}=0$$

Ex. 9. *Evaluate* $\iiint e^{x+y+z}\,dx\,dy\,dz$ *taken over +ive octant, such that* $x+y+z\leqslant 1$. **(Rajputana 55)**

Here in +ive octant x, y, z are all +ive.

$$\therefore \quad 0<x+y+z\leqslant 1.$$

Hence proceeding exactly as in *Ex.* 8,

$$I=\frac{1}{\Gamma 3}\int_0^1 h^2.e^h.dh.$$

Integrating successively by parts,

$$=\frac{1}{2}\left[e^h.(h^2-2h+2)\right]_0^1=\tfrac{1}{2}[e-2]=\tfrac{1}{2}e-1.$$

Ex. 6. *Find the value of* $\iint\ldots\int dx_1\, dx_2 \ldots dx_n$ *extended to all positive values of the variables, subject to the condition* $x_1^2+x_2^2+\ldots+x_n^2 < R^2$. **(Agra 67)**

Put $\left(\frac{x_1}{R}\right)^2=u_1$ or $x_1=Ru_1^{1/2}$; $\therefore\ dx_1=\frac{R}{2}\, u_1^{\frac{1}{2}-1}\, du_1$.

Also $u_1+u_2+\ldots+u_n < 1$.

$$\therefore\ I=\left(\frac{R}{2}\right)^n \iint\ldots\int u_1^{\frac{1}{2}-1}\, u_2^{\frac{1}{2}-1}\ldots u_n^{\frac{1}{2}-1}.du_1.du_2\ldots du_n$$

$$=\frac{R^n}{2^n}\,\frac{\Gamma\frac{1}{2}.\Gamma\frac{1}{2}.\ldots\Gamma\frac{1}{2}}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\ldots\frac{1}{2})}=\frac{R^n}{2^n}.\frac{(\sqrt{\pi})^n}{\Gamma\left(1+\frac{n}{2}\right)}.$$

Ex. 7. *Prove that the area in the positive quadrant between the curve* $x^n+y^n=a^n$ *and the axes is*

$$\frac{a^2}{2n}\,\frac{[\Gamma(1/n)]^2}{\Gamma(2/n)}$$ **(Gujrat 52)**

$\left(\frac{x}{a}\right)^n=u$, $\therefore\ x=au^{1/n},\ dx=\frac{a}{n}\, u^{(1/n)-1}\, du.$

$$\therefore\ \text{area}=\iint dx\, dy=\frac{a^2}{n^2}.\iint u^{(1/n)-1}\, v^{(1/n)-1} du\, dv,$$

where $u+v \leqslant 1$

$$=\frac{a^2}{n^2}.\frac{\Gamma\left(\frac{1}{n}\right).\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(1+\frac{2}{n}\right)}=\frac{a^2}{n^2}\,\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)}=\frac{a^2}{2n}.\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}.$$

Now we shall give some examples based on Liouville's extension of Dirichlet's Theorem, *i.e.*

$$\iiint F(x+y+z)\, x^{l-1}y^{m-1}z^{n-1}\, dx\, dy\, dz$$

$$=\frac{\Gamma l.\Gamma m\,\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} F(h)\, h^{l+m+n-1}\, dh,$$

where $h_1 < (x+y+z) < h_2$

Ex. 8. *Find the value of* $\iiint \log(x+y+z)\,dx\,dy\,dz$, *the integral extending over all +ive values of x, y, z, subject to the condition* $x+y+z<1$ **(Agra 45, 54, 57)**

Here x, y, z are all +ive and their sum is less than 1, *i.e.* $0<(x+y+z)<1$.

$$\therefore \quad I=\iiint \log(x+y+z)\,x^{1-1}y^{1-1}z^{1-1}\,dx\,dy\,dz$$

$$=\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1)}\int_0^1 \log(h)\,h^{1+1+1-1}\,dh$$

$$=\frac{1}{\Gamma 3}\int_0^1 h^2\log h\,dh$$

$$=\frac{1}{2}\left[\frac{h^3}{3}\log h-\frac{1}{3}\int h^2\,dh\right]$$

$$=\frac{1}{2}\left[\frac{h^3}{3}\log h-\frac{h^3}{9}\right]_0^1=-\frac{1}{18}.$$

$$\underset{x\to 0}{\text{Lt}}\,x^3\log x=0\times\infty=\underset{x\to 0}{\text{Lt}}\,\frac{\log x}{1/x^3}=\underset{x\to 0}{\text{Lt}}\,\frac{1/x}{-3/x^4}$$

$$=\underset{x\to 0}{\text{Lt}}\,\frac{-x^3}{3}=0.$$

Ex. 9. *Evaluate* $\iiint e^{x+y+z}\,dx\,dy\,dz$ *taken over +ive octant, such that* $x+y+z\leqslant 1$ **(Rajputana 55)**

Here in +ive octant x, y, z are all +ive.

$$\therefore \quad 0<x+y+z\leqslant 1.$$

Hence proceeding exactly as in *Ex.* 8,

$$I=\frac{1}{\Gamma 3}\int_0^1 h^2.e^h.dh.$$

Integrating successively by parts,

$$=\frac{1}{2}\left[e^h.(h^2-2h+2)\right]_0^1=\tfrac{1}{2}[e-2]=\tfrac{1}{2}e-1.$$

Ex. 6. *Find the value of* $\iint \ldots \int dx_1\, dx_2 \ldots dx_n$ *extended to all positive values of the variables, subject to the condition* $x_1^2+x_2^2+\ldots+x_n^2 < R^2$ **(Agra 67)**

Put $\left(\frac{x_1}{R}\right)^2 = u_1$ or $x_1 = Ru_1^{1/2}$, $\therefore\ dx_1 = \frac{R}{2} u_1^{\frac{1}{2}-1}\, du_1$.

Also $u_1+u_2+\ldots+u_n < 1$

$$\therefore\ I = \left(\frac{R}{2}\right)^n \iint \ldots \int u_1^{\frac{1}{2}-1}\, u_2^{\frac{1}{2}-1} \ldots u_n^{\frac{1}{2}-1} . du_1\, du_2 \ldots du_n$$

$$= \frac{R^n}{2^n} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2} \ldots \Gamma\frac{1}{2}}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2})} = \frac{R^n}{2^n} \frac{(\sqrt{\pi})^n}{\Gamma\left(1+\frac{n}{2}\right)}.$$

Ex. 7. *Prove that the area in the positive quadrant between the curve* $x^n+y^n=a^n$ *and the axes is*

$$\frac{a^2}{2n} \frac{[\Gamma(1/n)]^2}{\Gamma(2/n)}$$ **(Gujrat 52)**

$\left(\frac{x}{a}\right)^n = u$, $\therefore\ x = au^{1/n}$, $dx = \frac{a}{n} u^{(1/n)-1}\, du$

$\therefore$ area $= \iint dx\, dy = \frac{a^2}{n^2} . \iint u^{(1/n)-1}\, v^{(1/n)-1} du\, dv$,

where $u+v \leqslant 1$

$$= \frac{a^2}{n^2} . \frac{\Gamma\left(\frac{1}{n}\right)\,\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(1+\frac{2}{n}\right)} = \frac{a^2}{n^2} \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)} = \frac{a^2}{2n} . \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}.$$

Now we shall give some examples based on Liouville's extension of Dirichlet's Theorem, *i.e.*

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma l . \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(h)\, h^{l+m+n-1}\, dh,$$

where $h_1 < (x+y+z) < h_2$

Ex. 8. *Find the value of* $\iiint \log (x+y+z)\, dx\, dy\, dz$, *the integral extending over all +ive values of x, y, z, subject to the condition* $x+y+z < 1$ **(Agra 45, 54, 57)**

Here x, y, z are all +ive and their sum is less than 1, i.e. $0 < (x+y+z) < 1$

$$\therefore \quad I = \iiint \log (x+y+z)\, x^{1-1} y^{1-1} z^{1-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma 1\ \Gamma 1\ \Gamma 1}{\Gamma(1+1+1)} \int_0^1 \log (h) . h^{1+1+1-1}\, dh$$

$$= \frac{1}{\Gamma 3} \int_0^1 h^2 \log h\, dh$$

$$= \frac{1}{2} \left[\frac{h^3}{3} \log h - \frac{1}{3} \int h^2\, dh \right]$$

$$= \frac{1}{2} \left[\frac{h^3}{3} \log h - \frac{h^3}{9} \right]_0^1 = -\frac{1}{18}.$$

$$\underset{x \to 0}{\text{Lt}}\ x^3 \log x = 0 \times \infty = \underset{x \to 0}{\text{Lt}}\ \frac{\log x}{1/x^3} = \underset{x \to 0}{\text{Lt}}\ \frac{1/x}{-3/x^4}$$

$$= \underset{x \to 0}{\text{Lt}}\ \frac{-x^3}{3} = 0.$$

Ex. 9. *Evaluate* $\iiint e^{x+y+z}\, dx\, dy\, dz$ *taken over +ive octant, such that* $x+y+z \leqslant 1$. **(Rajputana 55)**

Here in +ive octant x, y, z are all +ive.

$$\therefore \quad 0 < x+y+z \leqslant 1.$$

Hence proceeding exactly as in *Ex.* 8,

$$I = \frac{1}{\Gamma 3} \int_0^1 h^2 . e^h . dh.$$

Integrating successively by parts,

$$= \frac{1}{2} \left[e^h . (h^2 - 2h + 2) \right]_0^1 = \tfrac{1}{2} [e-2] = \tfrac{1}{2} e - 1.$$

Ex. 10. *Evaluate* $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$ *for all +ive values of the variables subject to the condition*

$$x+y+z \leqslant \frac{\pi}{2}. \qquad \textbf{(Agra 63, 53)}$$

Here also $\quad 0 < x+y+z \leqslant \frac{\pi}{2}.$

$$\therefore \quad I = \iiint \sin(x+y+z)\, x^{2-1} y^{2-1} z^{2-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma 2\ \Gamma 2\ \Gamma 2}{\Gamma(2+2+2)} \cdot \int_0^{\pi/2} \sin h \,.\, h^{2+2+2-1}\, dh$$

$$= \frac{1}{\Gamma 6} \int_0^{\pi/2} h^5 \sin h\, dh.$$

Integrating successively by parts,

$$= \frac{1}{5\,!} \Big[h^5 (-\cos h) - (5h^4)(-\sin h) + (20h^3)(\cos h)$$

$$-(60h^2)(\sin h) + (120h)(-\cos h) - 120(-\sin h) \Big]_0^{\pi/2}.$$

All the above vanish for $h=0$ and all those which involve $\cos h$ vanish for $h=\frac{\pi}{2}$.

$$\therefore \quad I = \frac{1}{120} \left[-5 \left(\frac{\pi}{2}\right)^4 (-1) - 60 \left(\frac{\pi}{2}\right)^2 .(1) - 120(-1) \right]$$

$$= \frac{1}{120} \left[\frac{5\pi^4}{16} - 15\pi^2 + 120 \right]$$

or $\qquad I = \frac{1}{384} [\pi^4 - 48\pi^2 + 384].$

Ex. 11. *Prove that* $I = \iiiint dx\, dy\, dz\, dw$ *for all values of the variables for which* $x^2+y^2+z^2+w^2$ *is not less than* a^2 *and not greater than* b^2 *is* $\frac{\pi^2}{32}(b^4-a^4)$. **(Punjab 60)**

Here we are given that $a^2 < x^2+y^2+z^2+u^2 < b^2$.

In order to reduce to standard form put $x^2 = u_1$.

$\therefore \quad x=\sqrt{u_1}$ and $dx=\frac{1}{2}u_1^{-\frac{1}{2}}\,du=\frac{1}{2}u_1^{\frac{1}{2}-1}du$.

$$\therefore \quad a^2 < u_1+u_2+u_3+u_4 < b^2.$$

$$I=\frac{1}{16}\iiiint u_1^{\frac{1}{2}-1}u_2^{\frac{1}{2}-1}\,u_3^{\frac{1}{2}-1}\,u_4^{\frac{1}{2}-1}\,du_1\,du_2\,du_3\,du_4$$

$$=\frac{1}{16}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_{a^2}^{b^2} h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dh$$

by Liouville's theorem

$$=\frac{1}{16}\,\frac{(\sqrt{\pi})^4}{\Gamma 2}\int_{a^2}^{b^2}h\,dh=\frac{\pi^2}{32}\,(b^4-a^4) \qquad \textbf{Proved.}$$

Ex. 12. *Prove that when $x+y<h$ and x, y are +ive,*

$$\iint F'(x+y)\,x^{l-1}\,y^{-l}\,dx\,dy=\frac{\pi}{\sin l\pi}\,[F(h)-F(0)].$$

When $0 < x+y < h$, we have by Liouville's theorem,

$$I=\iint F'(x+y)\,x^{l-1}\,y^{(1-l)-1}\,dx\,dy$$

$$=\frac{\Gamma l\;\Gamma\,(1-l)}{\Gamma\,(l+1-l)}\int_0^h F'\,(h)\,h^{l+(1-l)-1}\,dh.$$

Now we know that $\Gamma(n)\;\Gamma(1-n)=\frac{\pi}{\sin n\pi}$. [P. 83]

$$\therefore \quad I=\frac{\pi}{\sin l\pi}\int_0^h F'(h)\,dh=\frac{\pi}{\sin l\pi}\Big[F(h)\Big]_0^h$$

$$=\frac{\pi}{\sin l\pi}\,[F(h)-F(0)].$$

Ex. 13. *Prove that* $\iint x^{l-1}\,y^{-l}\,e^{x+y}\,dx\,dy=\frac{\pi}{\sin l\pi}\,(e^h-1)$ *extended to all positive values subject to* $(x+y)<h$.

Proceed exactly as in Q. 12.

when extended for all +ive values of x, y, subjected to the condition $x+y<\infty$

Questions reducible to Liouville's Theorem

Ex. 19. *Evaluate the integral*

$$\iint\cdots\int\frac{dx_1\,dx_2\ldots dx_n}{\sqrt{(a^2-x_1^2-x_2^2\ldots x_n^2)}},$$

the integral being extended to all +ive values of the variables for which the expression is real. **(Indore 66 ; Agra 1959)**

The expression will be real if $a^2-x_1^2-x_2^2\ldots x_n^2>0$.

or $$0<x_1^2+x_2^2+\ldots+x_n^2<a^2$$

In order to reduce to the standard form put $x_1^2=u_1$.

$$\therefore\quad x_1=\sqrt{u_1}$$

$$\therefore\quad dx_1=\tfrac{1}{2}u_1^{\frac{1}{2}-1}\text{ etc.}$$

$$\therefore\quad I=\left(\frac{1}{2}\right)^n\iint\cdots\int\frac{u_1^{\frac{1}{2}-1}.u_2^{\frac{1}{2}-1}\ldots u_n^{\frac{1}{2}-1}}{[a^2-(u_1+u_2+\ldots+u_n)]^{1/2}}\,du_1\,du_2\ldots du_n$$

where $0<u_1+u_2+\ldots u_n<a^2$

$$=\left(\frac{1}{2}\right)^n\cdot\frac{\Gamma\frac{1}{2}.\Gamma\frac{1}{2}\ldots\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+\frac{1}{2}+\ldots\frac{1}{2})}\int_0^{a^2}\frac{h^{(n/2)-1}}{\sqrt{(a^2-h)}}\,dh$$

Put $h=a^2\sin^2\theta$

$$=\left(\frac{1}{2}\right)^n\cdot\frac{\pi^{n/2}}{\Gamma(n/2)}\int_0^{\pi/2}\frac{(a^2\sin^2\theta)^{(n/2)-1}}{\sqrt{(a^2-a^2\sin^2\theta)}}\,2a^2\sin\theta\cos\theta\,d\theta$$

$$=\frac{\pi^{n/2}}{2^n\Gamma(n/2)}\,2a^{n-1}\int_0^{\pi/2}\sin^{n-1}\theta\,d\theta$$

$$=\frac{\pi^{n/2}}{2^n\Gamma(n/2)}\cdot 2a^{n-1}\,\frac{\Gamma\left(\frac{n}{2}\right).\Gamma\frac{1}{2}}{2\Gamma\left(\frac{n+1}{2}\right)}$$

$$=a^{n-1}\frac{\pi^{n/2}}{2^n}\cdot\frac{\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{\pi^{(n+1)/2}.a^{n-1}}{2^n\Gamma\left(\frac{n+1}{2}\right)}.$$

Ex. 20. *Evaluate the integrals*

$$(i) \quad \iiint \frac{dx\, dy\, dz}{\sqrt{(a^2-x^2-y^2-z^2)}}$$

(Sagar 62 ; Agra 60, 52, 66 ; Rajputana 52, 63)

$$(ii) \quad \iiint \frac{dx\, dy\, dz}{\sqrt{(1-x^2-y^2-z^2)}},$$

(Vikram 63 ; Sagar 64 ; Agra 50, 65 ; Rajputana 50, 58 ; Pb. 52)

the integrals being extended to all +ive values of the variables for which the expression is real

Both the above are particular cases of Ex. 19. Putting $n=3$, we get

$$I=\frac{\pi^2 a^2}{8\,\Gamma 2}=\frac{\pi^2 a^2}{8}.$$

Again putting $a=1$ for second integral, $I=\frac{\pi^2}{8}$.

Ex. 21. *Evaluate*

$$\iint \sqrt{\left(\frac{1-x^2/a^2-y^2/b^2}{1+x^2/a^2+y^2/b^2}\right)}\, dx\, dy, \text{ where } \frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1.$$

(Gauhati Hons. 67 ; Cal. Hons. 61)

In order to reduce to standard form, put $\frac{x^2}{a^2}=u$

or
$$x=a\sqrt{u}.$$

$$\therefore \quad dx=\frac{a}{2}\, u^{\frac{1}{2}-1}\, du.$$

$$\therefore \quad I=\iint \sqrt{\left\{\frac{1-(u+v)}{1+(u+v)}\right\}}\, \frac{a}{2}\, u^{\frac{1}{2}-1}\cdot\frac{b}{2}\, v^{\frac{1}{2}-1}\, du\, dv,$$

where $0 < u+v \leqslant 1$

$$=\frac{ab}{4}\,\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+\frac{1}{2})}\int_0^1 \sqrt{\left(\frac{1-h}{1+h}\right)}\, h^{\frac{1}{2}+\frac{1}{2}-1}\, dh$$

$$=\frac{ab}{4}\cdot\pi\int_0^1 \frac{1-h}{\sqrt{(1-h^2)}}\, dh=\frac{\pi ab}{4}\int_0^{\pi/2} (1-\sin\theta)\, d\theta,$$

where $h=\sin\theta$

$$=\frac{\pi ab}{4}\Big[\theta+\cos\theta\Big]_0^{\pi/2}=\frac{\pi ab}{4}\left[\frac{\pi}{2}-1\right].$$

Ex. 22. *Evaluate*

$$\iiint \sqrt{(a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2)}\, dx\, dy\, dz$$

taken throughout the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. **(Punjab 54)**

Let us consider the integral first within the positive octant and then for throughout the ellipsoid multiply by 8.

$$I=abc\iiint \sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)}\, dx\, dy\, dz.$$

Put $\dfrac{x^2}{a^2}=u$ as in Ex. 21.

$$\therefore\quad I=abc\,\frac{abc}{8}\iiint \sqrt{\{1-(u+v+w)\}}\,.u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw,$$

where $0<u+v+w<1$

$$=\frac{a^2b^2c^2}{8}\cdot\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 (1-h)^{1/2}\,h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dh$$

$$=\frac{a^2b^2c^2}{8}\frac{\pi^{3/2}}{\Gamma(\frac{3}{2})}\int_0^1 h^{\frac{3}{2}-1}\,(1-h)^{\frac{3}{2}-1}\,dh$$

$$=\frac{a^2b^2c^2}{8}\frac{\pi^{3/2}}{\Gamma(\frac{3}{2})}\cdot\frac{\Gamma\frac{3}{2}\,.\,\Gamma\frac{3}{2}}{\Gamma 3}=\frac{a^2b^2c^2\pi^{3/2}}{8}\cdot\frac{\frac{1}{2}\sqrt{\pi}}{2.1}=\frac{\pi^2a^2b^2c^2}{32}.$$

Ex. 23. *Evaluate* $\iiint \sqrt{\left(\dfrac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}\,dx\,dy\,dz$, *integral being taken over all positive values of x, y, z such that* $x^2+y^2+z^2\leqslant 1$. **(Rajputana 65, Punjab 56)**

Putting $x^2=u$ etc., we get

$$I=\frac{1}{8}\iiint \sqrt{\left\{\frac{1-(u+v+w)}{1+(u+v+w)}\right\}}.u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}.du\,dv\,dw,$$

where $\qquad 0<u+v+w\leqslant 1$

or $\qquad I=\dfrac{1}{8}\cdot\dfrac{(\Gamma\frac{1}{2})^3}{\Gamma\frac{3}{2}}\displaystyle\int_0^1\sqrt{\left(\frac{1-h}{1+h}\right)}.h^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,dh$

$$=\frac{\pi}{4}\int_0^1 \frac{1-h}{\sqrt{(1-h^2)}}\cdot\sqrt{h}\,dh.$$

Put $h^2=z$ or $h=\sqrt{z}$. $\therefore$ $dh=\frac{1}{2}z^{-1/2}\,dz$.

$$\therefore \quad I=\frac{\pi}{4}\int_0^1 (1-z)^{-1/2}\,[1-z^{1/2}]\,z^{1/4}.\tfrac{1}{2}z^{-1/2}\,dz$$

$$=\frac{\pi}{4}\int_0^1 (1-z)^{-1/2}\,(z^{-1/4}-z^{1/4})\,dz$$

$$=\frac{\pi}{4}\int_0^1 \left[z^{\frac{3}{4}-1}\,(1-z)^{\frac{1}{2}-1}-z^{\frac{5}{4}-1}.(1-z)^{\frac{1}{2}-1}\right]dz$$

$$=\frac{\pi}{8}\,[B(\tfrac{3}{4},\tfrac{1}{2})-B(\tfrac{5}{4},\tfrac{1}{2})]$$

You may put the answer in the form of Gamma function by replacing $B(m, n)$ by $\dfrac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$.

Ex. 24. *Prove that*

$$\iint_D e^{-x^2-y^2}\,dx\,dy=\frac{\pi}{4}\,(1-e^{-R^2})$$

where D is the region defined by $x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant R^2$.

(Gujrat 52)

Put $x^2=u$ etc. $\therefore$ $0 < u+v < R^2$.

$$\therefore \quad I=\tfrac{1}{4}\iint e^{-(u+v)}u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}\,du\,dv$$

$$=\tfrac{1}{4}.\frac{\Gamma\frac{1}{2}.\Gamma\frac{1}{2}}{\Gamma 1}\int_0^{R^2} e^{-h}.h^{\frac{1}{2}+\frac{1}{2}-1}\,dh$$

$$=\frac{\pi}{4}\Big[-e^{-h}\Big]_0^{R^2}=\frac{\pi}{4}\,[1-e^{-R^2}].$$

Ex. 25. *With certain restrictions on the values of a, b, m and n, prove that*

$$\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\,x^{2m-1}.y^{2n-1}\,dx\,dy=\frac{\Gamma m\,\Gamma n}{4a^m b^n}.$$

Let us put $ax^2=u$, $by^2=v$.

$$\therefore \quad x=\frac{1}{\sqrt{a}}\sqrt{u} \text{ and } dx=\frac{1}{2\sqrt{a}}\, u^{\frac{1}{2}-1}\, du \text{ etc.}$$

Also u and v are all to be +ive and hence a and b should be +ive

$$\therefore \quad 0 < u+v < \infty.$$

$$\therefore \quad I=\iint \frac{e^{-(u+v)}}{4a^m b^n}\, u^{m-1}\, v^{n-1}\, du\, dv$$

or
$$I=\frac{1}{4a^m b^n}\cdot\frac{\Gamma m.\Gamma n}{\Gamma(m+n)}\int_0^\infty e^{-h}.h^{m+n-1}\, dh.$$

Now
$$\Gamma n=\int_0^\infty e^{-x}\, x^{n-1}\, dx.$$

$$\therefore \quad I=\frac{1}{4a^m b^n}\cdot\frac{\Gamma m\Gamma n}{\Gamma(m+n)}\cdot\Gamma(m+n)=\frac{\Gamma m.\Gamma n}{4a^m.b^n}. \quad \textbf{Proved.}$$

CHAPTER IV

VOLUMES AND SURFACES

§ 1. ***Formula for volume.*** The volume of a solid is given by the formula $\iiint dx\,dy\,dz$ where the limits of integration are to be so chosen as to include the whole required volume

We know that mass = volume × density.

$$= \iiint \rho\, dx\, dy\, dz.$$

Centroid of mass $\bar{x} = \dfrac{\iiint x.\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$ **and similar expressions for** y **and** z.

Moment of inertia of a body about a line is Σmr^2 **where m is the mass of an element and r is perpendicular distance of the element from the line.**

We shall use the above formula in calculating the volumes of solids Firstly we shall give some questions in which the integration is reduced to Dirichlet's form to be followed by questions in which actual integration has to be performed.

Exercises

Ex. 1. *Find the volume of ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \textbf{(Punjab 52 ; Vikram 64)}$$

Let us first find and the volume in the positive octant *and multiply it by 8 to get the entire volume.*

$$V = \iiint dx\, dy\, dz.$$

For points within +ive octant, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} < 1.$

Put $\frac{x^2}{a^2} = u$ or $x = a\sqrt{u}$. $\therefore$ $dx = \frac{a}{2} u^{\frac{1}{2}-1}\, du$ etc.

$\therefore$ $V = \frac{abc}{8}\iiint u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw$

where $u+v+w \leqslant 1$

$$= \frac{abc}{8} \cdot \frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} = \frac{abc}{8} \cdot \frac{(\sqrt{\pi})^3}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{\pi abc}{6}.$$

(Ex. 5 P. 109)

$\therefore$ total volume $= 8\cdot\frac{\pi abc}{6} = \frac{4}{3}\pi abc$

Ex. 2. (a) *Find the mass of an octant of ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

where the density at any point is $\rho = kxyz$. **(Rajputana 57)**

Mass $= \iiint \rho\, dx\, dy\, dz = k\iiint xyz\, dx\, dy\, dz$.

Proceeding as above by putting $\frac{x^2}{a^2} = u$ etc as in Ex. 1,

$$M = \frac{ka^2b^2c^2}{8}\iiint du\, dv\, dw$$

$$= k\frac{a^2b^2c^2}{8}\iiint u^{1-1}v^{1-1}w^{1-1}\, du\, dv\, dw$$

$$= \frac{k}{8}a^2b^2c^2\,\frac{\Gamma 1\;\Gamma 1\;\Gamma 1}{\Gamma(1+3)} = \frac{ka^2b^2c^2}{48}, \quad \because \Gamma 4 = 3\,! = 6$$

(b) *The space enclosed by the surface*

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3 = 1$$

is full of matter whose density at any point is given by $\rho = (xyz)^3$. *Find the whole mass.* **(Sagar 64)**

Proceed as in part (a).

Ex. 3. *Find the centroid of an octant of a uniform ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$. **(Agra 56)**

$$\therefore \quad \bar{x} = \frac{\iiint x.\rho\, dx\, dy\, dz}{\iiint \rho.dx\, dy\, dz} = \frac{\iiint x.dx\, dy\, dz}{\iiint dx\, dy\, dz}.$$

The denominator as calculated in Ex. 1 is $\frac{\pi abc}{6}$.

Now proceeding as in Ex. 1, the numerator reduces to

$$\frac{a^2bc}{8}\iiint u^{1-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\, du\, dv\, dw, \quad \text{where } u+v+w \leqslant 1$$

$$=\frac{a^2bc}{8}\cdot\frac{\Gamma 1.\Gamma\frac{1}{2}.\Gamma\frac{1}{2}}{\Gamma(1+1+\frac{1}{2}+\frac{1}{2})}=\frac{a^2bc}{8}\cdot\frac{\pi}{\Gamma 3}=\frac{\pi a^2bc}{16}.$$

$$\therefore \quad \bar{x}=\frac{\pi a^2bc}{16}\div\frac{\pi abc}{6}=\frac{3}{8}a.$$

Similarly, $\bar{y}=\frac{3}{8}b,\ \bar{z}=\frac{3}{8}c$

Ex. 4. *Find the volume enclosed by the surface*

$$\left(\frac{x}{a}\right)^{2n}+\left(\frac{y}{b}\right)^{2n}+\left(\frac{z}{c}\right)^{2n}=1$$

(Agra 61, 66 ; Rajputana 56)

Let us calculate the volume in the + ive octant.

$$V=\iiint dx\, dy\, dz.$$

Put $\left(\frac{x}{a}\right)^{2n}=u$; $\therefore \quad x=au^{1/2n},\ dx=\frac{a}{2n}u^{(1/2n)-1}\,du.$

$$\therefore \quad V=\frac{abc}{8n^3}\iiint u^{(1/2n)-1}.v^{(1/2n)-1}.w^{(1/2n)-1}\,du\,dv\,dw$$

$$=\frac{abc}{8n^3}\frac{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(1+\frac{1}{2n}+\frac{1}{2n}+\frac{1}{2n}\right)}=\frac{abc}{8n^3}\cdot\frac{\left(\Gamma\frac{1}{2n}\right)^3}{\Gamma\left(1+\frac{3}{2n}\right)}$$

$$=\frac{abc}{8n^3}\cdot\frac{\left(\Gamma\frac{1}{2n}\right)^3}{\frac{3}{2n}.\Gamma\frac{3}{2n}}=\frac{abc}{12n^2}\frac{\left(\Gamma\frac{1}{2n}\right)^3}{\Gamma\left(\frac{3}{2n}\right)}. \qquad \ldots(1)$$

$$\text{Hence the total volume}=8V=\frac{2}{3}\frac{abc}{n^2}\frac{\left(\Gamma\frac{1}{2n}\right)^3}{\Gamma\left(\frac{3}{2n}\right)}. \qquad \ldots(2)$$

Ex. 5. *Find the volume bounded by the positive sides of the co-ordinate planes and the surface*

$$\left(\frac{x}{a}\right)^{1/2}+\left(\frac{y}{b}\right)^{1/2}+\left(\frac{z}{c}\right)^{1/2}=1$$ **(Vikram 63 ; Agra 48)**

Proceed exactly as in Ex. 4 or putting $2n=\frac{1}{2}$, *i.e.* $n=\frac{1}{4}$ in (1), we get

$$V=\text{volume in +ive octant}$$
$$=\frac{abc}{12\left(\frac{1}{16}\right)}\cdot\frac{(\Gamma 2)^3}{\Gamma 6}=\frac{4abc}{3}\cdot\frac{1}{5!}$$
$$=\frac{4abc}{3}\cdot\frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1}=\frac{abc}{90}.$$

Ex. 6. (a) *Find the entire volume of the solid bounded by the surface* $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}+\left(\frac{z}{c}\right)^{2/3}=1$ **(Agra 53, 59)**

Proceed exactly as in Ex. 4 or putting $2n=\frac{2}{3}$, *i.e.* $n=\frac{1}{3}$ in (2) of Ex. 4 for entire volume, we get

$$\text{volume}=\frac{2}{3}\,\frac{abc}{\left(\frac{1}{3}\right)^2}\,\frac{\left(\Gamma\frac{3}{2}\right)^3}{\Gamma\frac{9}{2}}=\frac{6abc\,.\left(\frac{1}{2}\sqrt{\pi}\right)^3}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}=\frac{4}{35}\,abc\pi.$$

(b) *Prove that the centroid of the portion of the above surface in the first octant is* $\left[\frac{21a}{128},\frac{21b}{128},\frac{21c}{128}\right]$.

Ex. 7. *Find the volume of the tetrahedron bounded by co-ordinate planes and the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, *where* a, b, c *are* $+$*ve.* **(Gauhati Hons. 65)**

Proceed as in Ex. 4 or put $2n=1$ or $n=\frac{1}{2}$ in (1) of Ex. 4 for volume in $+$ive octant.

$$\therefore\quad V=\frac{abc}{12\left(\frac{1}{2}\right)^2}\cdot\frac{(\Gamma 1)^3}{\Gamma 3}=\frac{abc}{3\cdot 2}=\frac{abc}{6}.$$

Ex. 8. *Show that the volume of a solid the equation of whose surface is* $\frac{x^4}{a^4}+\frac{y^4}{b^4}+\frac{z^4}{c^4}=1$ *is* $\frac{abc}{12\pi}\sqrt{2}\,[\Gamma\frac{1}{4}]^4$. **(Agra 55)**

Proceed as in Ex. 4 or put $2n=4$ *i.e.* $n=2$ in (2) of Ex. 4 for total volume which is therefore equal to $\frac{2}{3}\cdot\frac{abc}{(2)^2}\frac{(\Gamma\frac{1}{4})^3}{\Gamma\frac{3}{4}}$.

In order to bring the required form of the answer, we multiply above and below by $\Gamma(\frac{1}{4})$

$$\therefore \quad V=\frac{abc}{6}\frac{(\Gamma\frac{1}{4})^4}{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}.$$

Now $\Gamma\frac{1}{4}.\Gamma\frac{3}{4}=\Gamma\frac{1}{4}.\Gamma(1-\frac{1}{4})=\dfrac{\pi}{\sin\frac{\pi}{4}}=\sqrt{2}\pi.$

$$\because \quad \Gamma n\;\Gamma(1-n)=\frac{\pi}{\sin n\pi} \text{ [R. 2 P. 83]},$$

$$\therefore \quad V=\frac{abc}{6\pi\sqrt{2}}.(\Gamma\tfrac{1}{4})^4=\frac{abc\sqrt{2}}{12\pi}\,[\Gamma\tfrac{1}{4}]^4. \qquad \textbf{Proved.}$$

Ex. 9. *Find the volume in the first octant determined by the surface $x^n+y^n+z^n=a^n$, $n>0$.*

The given surface is $\left(\frac{x}{a}\right)^n+\left(\frac{y}{a}\right)^n+\left(\frac{z}{a}\right)^n=1.$

Proceed as in Ex. 4 or replace $2n$ by n *i.e.* n by $n/2$ in *1* of Ex. 4 for volume in the +ive octant,

$$V=\frac{abc}{12.\left(\frac{n}{2}\right)^2}\cdot\frac{\Gamma\left(\frac{1}{n}\right)^3}{\Gamma\left(\frac{3}{n}\right)}=\frac{abc\left[\Gamma\left(\frac{1}{n}\right)\right]^3}{3n^2.\Gamma\left(\frac{3}{n}\right)}.$$

It can also be put as $\dfrac{abc\left\{\Gamma\left(\frac{1}{n}\right)\right\}^3}{n^3.\frac{3}{n}\Gamma\left(\frac{3}{n}\right)}=\dfrac{abc\left\{\Gamma\left(\frac{1}{n}\right)\right\}^3}{n^3\Gamma\left(1+\frac{3}{n}\right)}.$

Ex. 10. *Find the whole volume of the solid bounded by the surface whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^4}{c^4}=1$.* **(Agra 51)**

$$V=8\iiint dx\ dy\ dz.$$

Put $\left(\frac{x}{a}\right)^2=u,\ \left(\frac{y}{b}\right)^2=v$ and $\left(\frac{z}{c}\right)^4=w$, etc.

$$\therefore\quad V=8\,\frac{abc}{16}\cdot\iiint u^{\frac12-1}v^{\frac12-1}w^{\frac14-1}\,du\ dv\ dw$$

where $u+v+w\leqslant 1$

$$=\frac{abc}{2}\,\frac{\Gamma\frac12\ \Gamma\frac12\ \Gamma\frac14}{\Gamma(1+\frac12+\frac12+\frac14)}=\frac{abc}{2}\cdot\frac{\pi.\Gamma\frac14}{\frac34.\frac14\Gamma\frac14}=\frac{8\pi abc}{5}.$$

Ex. 11. *Find the mass of the tetrahedron bounded by the coordinate planes and the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, *the variable density being* $\rho=kxyz$. **(Jiwaji 66 ; Agra 58)**

Mass in the +ive octant $=\iiint \rho.dx\ dy\ dz$

or
$$M=\iiint kxyz.dx\ dy\ dz.$$

Put $\frac{x}{a}=u$ etc.

$$\therefore\quad M=ka^2b^2c^2\iiint u^{2-1}v^{2-1}w^{2-1}.du\ dv\ dw$$

or
$$M=ka^2b^2c^2\,\frac{\Gamma2\Gamma2\Gamma2}{\Gamma(1+2+2+2)}=\frac{a^2b^2c^2.k}{6.5.4.3.2.1}=\frac{ka^2b^2c^2}{720}.$$

Ex. 12. (a) *Find the centroid of mass in +ive octant of the surface* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *whose density is given by*

$$\rho=\mu.x^p\ y^q.z^r$$

(i)
$$\bar{x}=\frac{\iiint x.\rho.dx\ dy\ dz}{\iiint \rho.dx\ dy\ dz}=\frac{\iiint \mu.x^{p+1}.y^q.z^r.dx\ dy dz}{\iiint \mu.x^p.y^q.z^r.dx\ dy\ dz}$$

The denominator here stands for the mass

Put $\left(\frac{x}{a}\right)^2=u$; $\therefore\ \ x=a\sqrt{u}$ and $dx=\frac{a}{2\sqrt{u}}\,du.$

$$\therefore \quad D^r = \mu \iiint a^p\, b^q . c^r . u^{p/2}\, v^{q/2} . w^{r/2} . \frac{abc}{8\sqrt{(uvw)}}\, du\, dv\, dw$$

$$= \frac{\mu}{8} a^{p+1} . b^{q+1}\; c^{r+1} \iiint u^{\frac{p+1}{2}-1}\, v^{\frac{q+1}{2}-1}\, w^{\frac{r+1}{2}-1}\, du\, dv\, dw$$

$$= \frac{\mu}{8} a^{p+1} . b^{q+1}\; c^{r+1}\; \frac{\Gamma\left(\frac{p+1}{2}\right) . \Gamma\left(\frac{q+1}{2}\right) . \Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+5}{2}\right)}$$

Replacing p by $(p+1)$, we get the value of N^r as

$$\frac{\mu}{8} a^{p+2} . b^{q+1}\; c^{r+1} \frac{\Gamma\left(\frac{p+2}{2}\right) . \Gamma\left(\frac{q+1}{2}\right) . \Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+6}{2}\right)}$$

$$\therefore \quad \bar{x} = \frac{N^r}{D^r} = a \frac{\Gamma\left(\frac{p+2}{2}\right) \Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{p+q+r+6}{2}\right)}$$

Similarly we can find the values of $\bar{y}$ and $\bar{z}$. Otherwise also it is clear that $\frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c}$.

(ii) *If the density varies as* xyz, then putting $p=q=r=1$ we get $\frac{\bar{x}}{a} = \frac{16}{25} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c}$.

(iii) *If the density varies as* $x^2y^2z^2$, then putting $p=q=r=2$ we get
$$\frac{\bar{x}}{a} = \frac{67}{128} = \frac{\bar{y}}{b} = \frac{\bar{z}}{c}.$$

(iv) *If the density were constant*, then $p=q=r=0$.

(b) *Find the moment of inertia of a homogeneous ellipsoid of unit density about axis of* z. **(Rajputana 60)**

Let the mass of the element be $\rho\ \delta x\ \delta y\ \delta z$ and its distance from z-axis is $\sqrt{(x^2+y^2)}$.

$\therefore$ M.I. $=8\iiint \rho\,(x^2+y^2)\,dx\,dy\,dz$, the limits of integration being extended for +ive octant only

Now putting $\dfrac{x^2}{a^2}=u$ etc.,

$$\text{M.I.}=8\iiint \rho\,(a^2u+b^2v)\,\tfrac{1}{8}au^{-1/2}bv^{-1/2}cw^{-1/2}\,du\,dv\,dw$$

where $u+v+w \leqslant 1$

$$=abc\iiint \rho\,\{a^2u^{\frac{3}{2}-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}+b^2u^{\frac{1}{2}-1}v^{\frac{3}{2}-1}w^{\frac{1}{2}-1}\}\,du\,dv\,dw$$

$$=abc\rho\,(a^2+b^2)\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}.\Gamma\frac{1}{2}}{\Gamma(1+\frac{3}{2}+\frac{1}{2}+\frac{1}{2})}=abc\rho\,(a^2+b^2).\frac{\pi}{\frac{5}{2}.\frac{3}{2}}$$

$$=\frac{4abc\rho}{15}\,(a^2+b^2)=\frac{M}{5}\,(a^2+b^2)$$

$\because$ $M=\frac{4}{3}\pi abc\rho$ for an ellipsoid (Ex. 1).

Ex. 13. *Find the volume enclosed by the surfaces*

$$x^2+y^2=cz,\; x^2+y^2=2ax,\; z=0. \qquad \text{(Agra 56)}$$

Limits of z are from 0 to $\dfrac{x^2+y^2}{c}$ independent of z.

Limits of y are from $-\sqrt{(2ax-x^2)}$ to $+\sqrt{(2ax-x^2)}$ independent of y and z.

Limits of x are from 0 to $2a$, independent of x, y, z.

$V=\iiint dx\;dy\;dz$ within the above limits and first integration to be performed w.r.t. right hand variable,

$$\therefore\quad V=\iint\Big[z\Big]_0^{(x^2+y^2)/c}dx\,dy=\frac{1}{c}\int_0^{2a}\int_{-k}^{k}(x^2+y^2)\,dx\,dy,$$

where $\qquad k=\sqrt{(2ax-x^2)}$.

Now integrating w.r.t. y treating x as constant and limits of y are of the form $-k$ to k say $[k=\sqrt{(2ax-x^2)}]$.

Since x^2+y^2, when regarded as a function of y alone is an even function, and we know that

$$\int_{-k}^{k} f(y)\,dy=2\int_0^k f(y)\,dy,$$

when $f(y)$ is an even function of y.

$$\therefore\quad V=\frac{2}{c}\int_0^{2a}\int_0^k (x^2+y^2)\,dx\,dy=\frac{2}{c}\int_0^{2a}\left[x^2y+\frac{y^3}{3}\right]_0^k dx$$

$$=\frac{2}{c}\int_0^{2a}[x^2.\sqrt{(2ax-x^2)}+\tfrac{1}{3}(2ax-x^2)^{3/2}]\,dx$$

$$=\frac{2}{c}\int_0^{2a} x^{5/2}\sqrt{(2a-x)}+\frac{x^{3/2}}{3}(2a-x)^{3/2}\,dx.$$

Now put $x=2a\sin^2\theta$; $\therefore$ $dx=4a\sin\theta\cos\theta\,d\theta$ and limits are adjusted as 0 to $\pi/2$ for θ.

$$\therefore\quad V=\frac{2}{c}\int_0^{\pi/2}[(2a)^{5/2}\sin^5\theta\,(2a)^{1/2}\cos\theta$$

$$+\tfrac{1}{3}(2a)^{3/2}\sin^3\theta.(2a)^{3/2}.\cos^3\theta]\,4a\sin\theta\cos\theta\,d\theta$$

$$=\frac{2}{c}.(2a)^3\,4a\int_0^{\pi/2}(\sin^6\theta\cos^2\theta+\tfrac{1}{3}\sin^4\theta\cos^4\theta)\,d\theta$$

$$=64.\frac{a^4}{c}\left[\frac{5.3.1.1}{8.6.4.2}.\frac{\pi}{2}+\frac{1}{3}\left(\frac{3.1.3.1}{8.6.4.2}.\frac{\pi}{2}\right)\right]$$

$$=64.\frac{a^4}{c}.\frac{\pi}{2}\left(\frac{5}{128}+\frac{1}{3}.\frac{3}{128}\right)=\frac{3\pi a^4}{2c}.$$

Ex. 14. *Find the volume of the wedge intercepted between the cylinder $x^2+y^2=2ax$ and the planes $z=x$, $z=2x$.*

(Gujrat 59)

Limits of z are from x to $2x$ whereas those of y and x are same in as in Q. 13.

$$\therefore\quad V=\iiint dx\,dy\,dz=\iint\left[z\right]_x^{2x} dx\,dy$$

$$=\int_0^{2a}\int_{-k}^{k} x\,dx\,dy,\ \text{where } k=\sqrt{(2ax-x^2)}$$

or $$V=2\int_0^{2a}\int_0^k x\,dx\,dy=2\int_0^{2a}\left[xy\right]_0^k dx.$$

$$=2\int_0^{2a} x\sqrt{(2ax-x^2)}\,dx$$

$$=2\int_0^{2a} x^{3/2}\sqrt{(2a-x)}\,dx.$$

Put $x=2a\sin^2\theta$ as in Ex. 13

$$\therefore\quad V=2\int_0^{\pi/2}(2a)^{3/2}\sin^3\theta\,(2a)^{1/2}\cos\theta\,(4a\sin\theta\cos\theta\,d\theta)$$

$$=2\,(2a)^2\,4a\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$=32a^3\cdot\frac{3.1}{6}\frac{1}{4.2}\cdot\frac{\pi}{2}=\pi a^3.$$

Ex. 15. *Find the volume of the portion of the cylinder determined by the equations $x^2+y^2-2ax=0$, which is intercepted between the planes $z=x\tan\alpha$, $z=x\tan\beta$.*

Limits of z are from $x\tan\alpha$ to $x\tan\beta$ and limits for y and z are as in Ex. 13

$$V=\iiint dx\,dy\,dz=\iint\Big[z\Big]_{x\tan\alpha}^{x\tan\beta}dx\,dy$$

$$=(\tan\beta-\tan\alpha)\int_0^{2a}\int_{-k}^{k}x\,dx\,dy,\quad\text{where } k=\sqrt{(2ax-x^2)}.$$

Now it reduces to exactly the same integration as in Ex. 14 whose value is πa^3.

$$\therefore\quad V=(\tan\beta-\tan\alpha)\,\pi a^3.$$

Ex. 16. *Show that the volume common to the surfaces $y^2+z^2=4ax$ and $x^2+y^2=2ax$ is $\frac{2}{3}(3\pi+8)a^3$.*

(Agra 58 ; Rajputana 58)

Limits of z are from $-\sqrt{(4ax-y^2)}$ to $\sqrt{(4ax-y^2)}$ say $-l$ to $+l$.

Limits of y are from $-\sqrt{(2ax-x^2)}$ to $\sqrt{(2ax-x^2)}$ say $-k$ to $+k$.

Limits of x are from 0 to $2a$.

$$V=\iiint dx\,dy\,dz=\iint \Big[z\Big]_{-l}^{l} dx\,dy=\iint 2l\,dx\,dy$$

$$=2\int_0^{2a}\int_{-k}^{k}\sqrt{(4ax-y^2)}\,dx\,dy=2\times 2\int_0^{2a}\int_0^{k}\sqrt{(4ax-y^2)}\,dx\,dy.$$

$\because$ $\sqrt{(4ax-y^2)}$ is an even function of y and limits are $-k$ to k.

$$\therefore\quad V=4\int_0^{2a}\left[\frac{y}{2}\sqrt{(4ax-y^2)}+\frac{4ax}{2}\sin^{-1}\frac{y}{\sqrt{(4ax)}}\right]_0^k dx,$$

where $k=\sqrt{(2ax-x^2)}$

$$=4\int_0^{2a}\left[\frac{k}{2}\sqrt{(4ax-k^2)}+\frac{4ax}{2}\sin^{-1}\frac{k}{\sqrt{(4ax)}}\right]dx.$$

Now put for k.

$$V=4\int_0^{2a}\left[\frac{\sqrt{(2ax-x^2)}}{2}\sqrt{(2ax+x^2)}+2ax.\sin^{-1}\sqrt{\left(\frac{2ax-x^2}{4ax}\right)}\right]dx.$$

$$V=4\int_0^{2a}\left[\frac{x}{2}\sqrt{(4a^2-x^2)}+2ax.\sin^{-1}\sqrt{\left(\frac{2a-x}{4a}\right)}\right]dx.$$

Keeping in view the term $\sin^{-1}\sqrt{\left(\frac{2a-x}{4a}\right)}$, we put $x=2a\cos\theta$, so that

$$\sqrt{\left(\frac{2a-x}{4a}\right)}=\sqrt{\left\{\frac{2a\,(1-\cos\theta)}{4a}\right\}}$$

$$=\sqrt{\left(\frac{4a\sin^2\theta/2}{4a}\right)}=\sin\frac{\theta}{2}.$$

Also $dx=-2a\sin\theta\,d\theta$ and limits are adjusted as $\pi/2$ to 0.

$$\therefore\quad V=4\int_{\pi/2}^{0}[a\cos\theta.2a\sin\theta+(4a^2\cos\theta).\theta/2]$$

$$\times(-2a\sin\theta\,d\theta)$$

$$=16a^3\int_0^{\pi/2}(\cos\theta\sin^2\theta+\theta\sin\theta\cos\theta)\,d\theta$$

$$=16a^3\left[\frac{1}{3}+\left\{\frac{\theta \sin^2\theta}{2}\right\}_0^{\pi/2}-\frac{1}{2}\int_0^{\pi/2}\sin^2\theta.1\,d\theta\right]$$

$$=16a^3\left[\frac{1}{3}+\frac{\pi}{4}-\frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)\right]$$

$$=16a^3\left(\frac{8+3\pi}{24}\right)=\frac{2}{3}a^3\,(3\pi+8).$$

Ex. 17. *Prove that the volume enclosed by the cylinder* $x^2+y^2=2ax$, $z^2=2ax$ *is* $\frac{128}{15}a^3$. **(Gujrat 52)**

Limits of z are from $-\sqrt{(2ax)}$ to $\sqrt{(2ax)}$, *i.e.*, $-l$ to l, whereas the limits of y and x are as in Ex. 13.

$$V=\iiint dx\,dy\,dz=\iint\Big[z\Big]_{-l}^{l}dx\,dy=\iint 2l\,dx\,dy$$

$$=2\int_0^{2a}\int_{-k}^{k}\sqrt{(2ax)}\,dx\,dy=2\times2\int_0^{2a}\int_0^{k}\sqrt{(2ax)}\,dx\,dy$$

$$=2\int_0^{2a}\Big[\sqrt{(2ax)}\,y\Big]_0^k dx=4\int_0^{2a}\sqrt{(2ax)}.\sqrt{(2ax-x^2)}\,dx$$

$$=4\sqrt{(2a)}\int_0^{2a}x\sqrt{(2a-x)}\,dx.\quad \text{Put } x=2a\sin^2\theta \text{ etc.}$$

$$=4\sqrt{(2a)}\int_0^{\pi/2}2a\sin^2\theta\sqrt{(2a)}\cos\theta.4a\sin\theta\cos\theta\,d\theta$$

$$=64a^3\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta=64a^3\frac{2.1.1}{5.3.1}=\frac{128}{15}a^3.$$

Ex. 18. *Find the volume of the portion cut off from the cylinder which is determined by* $2x^2+y^2=2ax$ *and the planes* $z=mx$ *and* $z=nx$.

Limits of z are from mx to nx.

Limits of y are from $-\sqrt{(2ax-2x^2)}$ to $+\sqrt{(2ax-2x^2)}$, *i.e.*, $-k$ to $+k$.

Limits of x are from 0 to a,

$$V=\iiint dx\,dy\,dz=\iint\Big[z\Big]_{mx}^{nx}dx\,dy.$$

$$V=\int_0^a\int_{-k}^{k}(n-m)\ x\,dx\,dy=2\int_0^a\int_0^k(n-m)\,x\,dx\,dy$$

$$=2\,(n-m)\int_0^a x\Big[y\Big]_0^k dx=2\,(n-m)\int \text{[illegible]}$$

$$=2\,(n-m)\int_0^a x\sqrt{(2ax-2x^2)}\,dx$$

$$=2\sqrt{2}\,(n-m)\int_0^a x^{3/2}\sqrt{(a-x)}\,dx.$$

Now put $x=a\sin^2\theta$; $dx=2a\sin$ [illegible]

$$\therefore\quad V=2\sqrt{2}\,(n-m)\int_0^{\pi/2}a^{3/2}\sin^3\theta\sqrt{a}\ \text{[illegible]}\ d\theta$$

$$=4\sqrt{2}\ \ (n-m)\,a^3\int_0^{\pi/2}\sin^4\theta\ \text{[illegible]}$$

$$=4\sqrt{2}\,(n-m)\,a^3\,\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}$$

$$=\frac{\pi\sqrt{2}\,(n-m)\,a^3}{8}.$$

Ex. 19. *Find the* [illegible] $y^2=ax$, $x=3a$.

Limits of z are [illegible] to $+l$.

Limits of y are [illegible] $-k$ [illegible]

Limits of x are [illegible]

$$=4\int_0^{3a}\left[\frac{\sqrt{(ax)}}{2}\sqrt{(3ax)}+2ax\sin^{-1}\frac{1}{2}\right]dx$$

$$=4\int_0^{3a}\left(\frac{\sqrt{3}}{2}ax+2ax\frac{\pi}{6}\right)dx$$

$$=4\left(\frac{\sqrt{3}}{2}a+\frac{a\pi}{3}\right)\left[\frac{x^2}{2}\right]_0^{3a}=\frac{2}{3}(3\sqrt{3}+2\pi)\ a.\frac{9a^2}{2}$$

$$=3a^3(3\sqrt{3}+2\pi).$$

Ex. 20. *Find the volume cut from the surface* $\frac{y^2}{b}+\frac{z^2}{c}=2x$ *by a plane parallel to that of y-z at a distance v from it.*

Limits of z are

$$-\sqrt{\left(\frac{c}{b}\right)}\sqrt{(2bx-y^2)}\text{ to }\sqrt{\left(\frac{c}{b}\right)}\sqrt{(2bx-y^2)}$$

i.e. $-l$ to $+l$

Limits of y are $-\sqrt{(2bx)}$ to $+\sqrt{(2bx)}$, *i.e.* $-k$ to k.

Limits of x are 0 to a.

$$V=\iiint_{-l}^{l}dx\,dy\,dz=2\iint\left[z\right]_0^l dx\,dy$$

$$=2\int_0^a\int_{-k}^{k}\sqrt{\left(\frac{c}{b}\right)}\sqrt{(2bx-y^2)}\,dx\,dy$$

$$=2\sqrt{\left(\frac{c}{b}\right)}\times2\int_0^a\int_0^k\sqrt{(2bx-y^2)}\,dx\,dy$$

$$=4\sqrt{\left(\frac{c}{b}\right)}\int_0^a\int_0^{\sqrt{(2bx)}}\sqrt{(2bx-y^2)}\,dx\,dy,$$

$\because\ \sqrt{(2bx-y^2)}$ is an even function of y

$$=4\sqrt{\left(\frac{c}{b}\right)}\int_0^a\left[\frac{y}{2}\sqrt{(2bx-y^2)}+\tfrac{1}{2}.2bx\sin^{-1}\frac{y}{\sqrt{(2bx)}}\right]_0^{\sqrt{(2bx)}}dx$$

$$=4\sqrt{\left(\frac{c}{b}\right)}\int_0^a 0+bx.\frac{\pi}{2}\,dx=2\sqrt{(bc)}\,\pi\left[\frac{x^2}{2}\right]_0^a dx=\pi a^2\sqrt{(bc)}.$$

Ex. 21. *Find the volume of the right elliptic cylinder whose axis coincides with x-axis and altitude 2a, equation of the base*

being $$\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ x=0.$$

Limits of z are $-\frac{c}{b}\sqrt{(b^2-y^2)}$ to $\frac{c}{b}\sqrt{(b^2-y^2)}$,

i.e. $-l$ to $+l$.

Limits of y are $-b$ to $+b$

Limits of x are 0 to $2a$

$$V=\iiint_{-l}^{l} dx\,dy\,dz=2\iint\left[z\right]_0^l dx\,dy$$

$$=2\int_0^{2a}\int_{-b}^{b}\frac{c}{b}\sqrt{(b^2-y^2)}\,dx\,dy$$

$$=\frac{2c}{b}\times 2\int_0^{2a}\int_0^{b}\sqrt{(b^2-y^2)}\,dx\,dy.$$

$\because\ \sqrt{(b^2-y^2)}$ is an even function of y,

$$\therefore\quad V=\frac{4c}{b}\int_0^{2a}\left[\frac{y}{2}\sqrt{(b^2-y^2)}+\frac{1}{2}b^2\sin^{-1}\frac{y}{b}\right]_0^b dx$$

$$=\frac{4c}{b}\int_0^{2a}0+\frac{1}{2}b^2.\frac{\pi}{2}\,dx=\pi bc\left[x\right]_0^{2a}=2\pi abc.$$

Ex. 22. *Find the volume cut from the sphere* $x^2+y^2+z^2=a^2$ *by the cylinder* $x^2+y^2=ax$.

(Gauhati Hons. 67 ; Rajputana 63, 65 ; Agra 49, 57, 63, 67 ; Jiwaji 66 ; Punjab 53)

Limits of z are $-\sqrt{[a^2-(x^2+y^2)]}$ to $\sqrt{[a^2-(x^2+y^2)]}$,
i.e. $-l$ to l.

Limits of y are $-\sqrt{(ax-x^2)}$ to $\sqrt{(ax-x^2)}$, *i.e.* $-k$ to k.

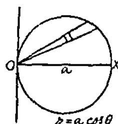
Limits of x are 0 to a.

$$V=\iiint_{-l}^{l} dx\,dy\,dz=2\int_0^a\int_{-k}^{k}\left[z\right]_0^l dx\,dy$$

$$=2\int_0^a\int_{-k}^{k}\sqrt{(a^2-x^2-y^2)}\,dx\,dy$$

$$=2\times 2\int_0^a\int_0^k\sqrt{(a^2-x^2-y^2)}\,dx\,dy. \qquad \ldots(1)$$

It will be very much convenient to change the integral to polar co-ordinates, *i.e.* the element $dx\,dy$ becomes $r\,d\theta\,dr$ and $x^2+y^2=r^2$.



$$V=4\iint \sqrt{(a^2-r^2)}\,.\,r\,d\theta\,dr.$$

Also $x^2+y^2=ax$ gives

$$r^2=a\,(r\cos\theta) \quad \text{or} \quad r=a\cos\theta\,; \quad \therefore \quad r=0 \text{ to } a\cos\theta$$

and limits of θ are from 0 to $\frac{\pi}{2}$.

$$\therefore \quad V=4\int_0^{\pi/2}\int_0^{a\cos\theta}\sqrt{(a^2-r^2)}\,r\,d\theta\,dr$$

$$=4\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{(a^2-r^2)^{3/2}}{\frac{3}{2}}\right]_0^{a\cos\theta}d\theta.$$

$$V=\frac{4}{3}\int_0^{\pi/2}[-a^3\sin^3\theta+a^3]\,d\theta$$

$$=\frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)=\frac{2}{3}a^3\left(\pi-\frac{4}{3}\right).$$

Another form. The question can also be put as under :

A sphere is cut by a right cylinder, the radius of whose base is half that of the sphere and one of whose edges passes through the centre of the sphere. Find the volume common to both. (Agra 54, 46, 33)

Let the equation of the sphere be $x^2+y^2+z^2=a^2$; then the equation of the cylinder whose one edge passes through the centre is $x^2+y^2=ax$.

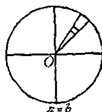
Ex. 23. *Find the volume cut from a sphere of radius a by a right circular cylinder with b as radius of the base and whose axis passes through the centre of the sphere.*

Let the equation of the sphere be $x^2+y^2+z^2=a^2$ and that of cylinder be $x^2+y^2=b^2$.

Limits of z are $-\sqrt{(a^2-x^2-y^2)}$ to $\sqrt{(a^2-x^2-y^2)}$,
i.e. $-l$ to $+l$.

Limits of y are $-\sqrt{(b^2-x^2)}$ to $\sqrt{(b^2-x^2)}$, i.e. $-k$ to k.

Limits of x are $-b$ to b



$$\therefore \quad V=\iiint_{-l}^{l} dx\, dy\, dz$$

$$=2\iint\Big[z\Big]_0^l\ dx\, dy$$

$$=2\int_{-b}^{b}\int_{-k}^{k}\sqrt{(a^2-x^2-y^2)}\ dx\, dy$$

$$=2\int_{-b}^{b}\int_{-\sqrt{(b^2-x^2)}}^{\sqrt{(b^2-x^2)}}\sqrt{(a^2-x^2-y^2)}\ dx\, dy$$

$$=8\int_0^b\int_0^{\sqrt{(b^2-x^2)}}\sqrt{(a^2-x^2-y^2)}\ dx\, dy.$$

Changing to polars $x=r\cos\theta$, $y=r\sin\theta$, $x^2+y^2=r^2=b^2$ and $dx\,dy=r\,d\theta\,dr$.

$$\therefore \quad V=8\int_0^{\pi/2}\int_0^b r\sqrt{(a^2-r^2)}\ d\theta\, dr$$

$$=8\int_0^{\pi/2}\left[-\frac{1}{2}\frac{(a^2-r^2)^{3/2}}{\frac{3}{2}}\right]_0^b\ d\theta$$

$$=-\frac{8}{3}\left[\{(a^2-b^2)^{3/2}-a^3\}\,\theta\ \right]_0^{\pi/2}=\frac{8}{3}\cdot\frac{\pi}{2}\cdot[a^3-(a^2-b^2)^{3/2}]$$

$$=\frac{4\pi}{3}\,[a^3-(a^2-b^2)^{3/2}].$$

Ex. 24. *The sphere $x^2+y^2+z^2=a^2$ is pierced by the cylinder $(x^2+y^2)^2=a^2(x^2-y^2)$. Prove that the volume of the sphere that lies inside the cylinder is*

$$\frac{8}{3}\left[\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right]a^3. \qquad \textbf{(Cal. Hons. 61)}$$

The limits of z are $-\sqrt{(a^2-x^2-y^2)}$ to $\sqrt{(a^2-x^2}$ say $-l$ to l.

$$=2\int_{-1}^{1}\int_{0}^{\sqrt{(1-x^2)}}[-(1-x^2)+y^2]\,dy$$

$$=2\int_{-1}^{1}\left[-(1-x^2)\,y+\frac{y^3}{3}\right]_0^{\sqrt{(1-x^2)}}dx$$

$$=2\int_{-1}^{1}-(1-x^2)^{3/2}+\frac{(1-x^2)^{3/2}}{3}\,dx$$

$$=2\left(-\frac{2}{3}\right)\int_{-1}^{1}(1-x^2)^{3/2}\,dx$$

$$=-\frac{4}{3}\times 2\int_{0}^{1}(1-x^2)^{3/2}\,dx$$

$$=-\frac{8}{3}\int_{0}^{\pi/2}\cos^3\theta\cos\theta\,d\theta,\ \text{where } x=\sin\theta$$

$$=-\frac{8}{3}\int_{0}^{\pi/2}\cos^4\theta\,d\theta=-\frac{8}{3}\cdot\frac{3.1}{4\ 2}\ \frac{\pi}{2}=-\frac{\pi}{2}=\frac{\pi}{2}.$$

Ex. 27. *Find the volume of the solid cut-off by the surface $z=(x+y)^2$ from the right prism whose base in the plane $z=0$ is the triangle bounded by the lines $x=0$, $y=0$, $x+y=1$.*

(Agra 61)

$$V=\int_0^1\int_0^{1-x}\int_0^{(x+y)^2}dx\,dy\,dz=\int_0^1\int_0^{1-x}(x+y)^2.dx\,dy$$

$$=\int_0^1\left[\frac{(x+y)^3}{3}\right]_0^{1-x}dx=\frac{1}{3}\int_0^1(1-x^3)\,dx$$

$$=\frac{1}{3}\left[x-\frac{x^4}{4}\right]_0^1=\frac{1}{3}\cdot\left(1-\frac{1}{4}\right)=\frac{1}{4}\ \text{units.}$$

Ex. 28. *Find the volume of the cylindrical column standing on the area common to the parabolas $y=x^2$, $x=y^2$ as base and cut-off by the surface $z=12+y-x^2$.*

Limits of z are 0 to $12+y-x^2$.

Limits of y are x^2 to $\sqrt{x}$.

Limits of x are 0 to 1.

The points of intersection of the parabolas $y=x^2$ and $x=y^2$ in the plane $z=0$ are $(0, 0)$ and $(1, 1)$.

Limits of z are 0 to $\frac{1}{x} \epsilon^{x+y}$

Limits of y are 0 to $1-x$ and limits of x are 0 to 1,

Elementary volume $\delta x\ \delta y\ \delta z$ enclosing the point (x, y, z) and the density is $\left(\frac{x}{y}\right)^{2/3}$.

$\therefore$ Mass of the element $= \left(\frac{x}{y}\right)^{2/3} \delta x\ \delta y\ \delta z.$

$$\therefore \text{ Total mass} = \int_0^1 \int_0^{1-x} \int_0^{(1/x)\ e^{x+y}} \left(\frac{x}{y}\right)^{2/3} dx\ dy\ dz$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{x}{y}\right)^{2/3} \Big[z \Big]_0^{(1/x)\ e^{x+y}} dx\ dy$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{x}{y}\right)^{2/3} \cdot \frac{1}{x} e^{x+y}\ dx\ dy$$

$$= \int_0^1 \int_0^{1-x} x^{-1/3} y^{-2/3} \epsilon^{x+y}\ dx\ dy$$

$$= \int_0^1 \int_0^{1-x} e^{x+y} x^{(2/3)-1} y^{(1/3)-1}\ dx\ dy,$$

where $0 < x+y \leqslant 1$.

Now we know that $\iiint F(x+y+z)\ x^{l-1} y^{m-1} z^{n-1}\ dx\ dy\ dz$

$$= \frac{\Gamma l\ \Gamma m\ \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(h) . h^{l+m+n-1}\ dh,$$

where $h_1 < (x+y+z) < h_2$.

$$\therefore \quad \text{Mass} = \frac{\Gamma(\frac{1}{3})\ \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+\frac{2}{3})} \int_0^1 e^h . h^{\frac{2}{3}+\frac{1}{3}-1}\ dh$$

$$= \frac{\Gamma(\frac{1}{3})\ \Gamma(1-\frac{1}{3})}{\Gamma 1} \Big[e^h \Big]_0^1 = \frac{\pi}{\sin \frac{\pi}{3}} (e-1) = \frac{2\pi}{\sqrt{3}} (e-1).$$

$\therefore$ $\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ and here $n = \frac{1}{3}$. (Page 83)

Limits of z are $-\sqrt{\left(c^2-\frac{a^2y^2}{x^2}\right)}$ to $+\sqrt{\left(c^2-\frac{a^2y^2}{x^2}\right)}$,

i.e. $-l$ to $+l$.

Limits of y are $-\frac{cx}{a}$ to $+\frac{cx}{a}$.

Limits of x are 0 to a.

$$V=2\iiint_{-l}^{l} dx\, dy\, dz=2\iint\Big[z\Big]_0^l dx\, dy$$

$$=2\int_0^a\int_{-cx/a}^{cx/a}\sqrt{\left(c^2-\frac{a^2y^2}{x^2}\right)}\, dx\, dy$$

$$=2\times 2\int_0^a\left[\int_0^{cx/a}\sqrt{\left(c^2-\frac{a^2y^2}{x^2}\right)}\, dy\right] dx. \qquad \ldots(1)$$

For integral w.r.t. y, put $\frac{ay}{x}=c\sin\theta$.

$$\therefore \quad \frac{a}{x}dy=c\cos\theta\, d\theta$$

and limits are adjusted as 0 to $\frac{\pi}{2}$.

$$\therefore \quad I=\int_0^{\pi/2} c\cos\theta.\frac{x}{a}.c\cos\theta\, d\theta=\frac{c^2}{a}\, x\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$=\frac{c^2}{a}x.\tfrac{1}{2}.\frac{\pi}{2}=\frac{\pi}{4}\frac{c^2}{a}x \qquad \text{Put in (1).}$$

$$\therefore \quad V=4\int_0^a\frac{\pi}{4}\frac{c^2}{a}\, x\, dx=\frac{\pi c^2}{a}\left[\frac{x^2}{2}\right]_0^a=\frac{\pi c^2}{a}.\frac{a^2}{2}=\frac{\pi ac^2}{2}.$$

Ex. 31. *The space enclosed by the planes $x=0$, $y=0$ and $x+y=1$ and the surface $zx=e^{x+y}$ is filled with matter whose density at any point (x, y, z) is given by $\rho=\left(\frac{x}{y}\right)^{2/3}$. Show that the whole mass is $\frac{2\pi}{\sqrt{3}}(e-1)$.*

(Agra 64, 54)

Limits of z are 0 to $\frac{1}{x} e^{x+y}$.

Limits of y are 0 to $1-x$ and limits of x are 0 to 1,

Elementary volume $\delta x\ \delta y\ \delta z$ enclosing the point (x, y, z) and the density is $\left(\frac{x}{y}\right)^{2/3}$.

$\therefore$ Mass of the element$=\left(\frac{x}{y}\right)^{2/3} \delta x\ \delta y\ \delta z$.

$$\therefore \text{ Total mass} = \int_0^1 \int_0^{1-x} \int_0^{(1/x)\, e^{x+y}} \left(\frac{x}{y}\right)^{2/3} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{x}{y}\right)^{2/3} \Big[z\Big]_0^{(1/x)\, e^{x+y}} dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \left(\frac{x}{y}\right)^{2/3} \cdot \frac{1}{x} e^{x+y}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} x^{-1/3} y^{-2/3} e^{x+y}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} e^{x+y} x^{(2/3)-1} y^{(1/3)-1}\, dx\, dy,$$

where $0 < x+y \leqslant 1$.

Now we know that $\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} F(h) . h^{l+m+n-1}\, dh,$$

where $h_1 < (x+y+z) < h_2$.

$$\therefore \quad \text{Mass} = \frac{\Gamma(\frac{1}{3})\, \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3}+\frac{2}{3})} \int_0^1 e^h . h^{\frac{1}{3}+\frac{2}{3}-1}\, dh$$

$$= \frac{\Gamma(\frac{1}{3})\, \Gamma(1-\frac{1}{3})}{\Gamma 1} \Big[e^h\Big]_0^1 = \frac{\pi}{\sin \frac{\pi}{3}} (e-1) = \frac{2\pi}{\sqrt{3}} (e-1).$$

$\therefore$ $\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ and here $n = \frac{1}{3}$. (Page 83)

Ex. 32. *Find the volume bounded by $cz=xy$, $z=0$, $y=b_1, b_2$; $x=a_1, a_2$.*

$$V=\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_0^{xy/c} dx\,dy\,dz=\int_{a_1}^{a_2}\int_{b_1}^{b_2}\frac{xy}{c}\,dx\,dy$$

$$=\int_{a_1}^{a_2}\frac{x}{c}\left[\frac{y^2}{2}\right]_{b_1}^{b_2} dx=\frac{b_2^2-b_1^2}{2c}\left[\frac{x^2}{2}\right]_{a_1}^{a_2}$$

$$=\frac{1}{4c}(a_2^2-a_1^2)(b_2^2-b_1^2).$$

Ex. 33. *Find the volume bounded by*

$$y^2=x+1,\ y^2=-x+1,\ z=-2,\ z=x+4.$$

Here let us choose the element as $\delta y\,\delta x\,\delta z$.

Limits of z are -2 to $x+4$.

Limits of x are y^2-1 to $1-y^2$ or $-(1-y^2)$ to $(1-y^2)$.

Limits of y are -1 to $+1$.

$$\therefore\quad V=\int_{-1}^{+1}\int_{-(1-y^2)}^{(1-y^2)}\int_{-2}^{x+4} dy\,dx\,dz$$

$$=\int_{-1}^{+1}\int_{-(1-y^2)}^{(1-y^2)}\Big[z\Big]_{-2}^{x+4} dy\,dx$$

$$=\int_{-1}^{+1}\int_{-(1-y^2)}^{1-y^2}(x+6)\,dy\,dx.$$

Now x is an odd function of x and the limits are of the form $-a$ to $+a$ and hence its integral vanishes whereas 6 is an even function of x.

$$\therefore\quad V=2\int_{-1}^{1}\int_0^{1-y^2} 6\,dy\,dx=2\int_{-1}^{1}\Big[6x\Big]_0^{1-y^2} dy$$

$$=12\int_{-1}^{1}(1-y^2)\,dy=12\times 2\int_0^1(1-y^2)\,dy$$

$$=24\left[y-\frac{y^3}{3}\right]_0^1 dy=24\left[1-\frac{1}{3}\right]=\frac{48}{3}.$$

Ex. 24. *The curve $z\,(a^2+x^2)^{3/2}=a^4$ lying in the $z-x$ plane revolves about the axis of z. Prove that the volume*

included in the +ve octant by the surface and $x=0$, $x=a$, $y=0$ and $y=a$ is $\pi a^3/6$.

Here we are given the equation of the curve in the zx plane which is revolved about z-axis. Hence replacing x by $\sqrt{(x^2+y^2)}$ in the given equation, the surface becomes

$$z\,(a^2+x^2+y^2)^{3/2}=a^4. \quad \text{(Note).}$$

$\therefore$ Limits of z are 0 to $\dfrac{a^4}{(a^2+x^2+y^2)^{3/2}}$ or k

and limits of x and y are 0 to a as given.

$$\therefore \quad V=\int_0^a\int_0^a\int_0^k dx\,dy\,dz=\int_0^a\int_0^a\Big[\,z\,\Big]_0^k dx\,dy$$

$$=\int_0^a\int_0^a\frac{a^4}{(a^2+x^2+y^2)^{3/2}}\,dx\,dy$$

Now we have to integrate w.r.t. y treating x as constant.

Put $y=\sqrt{(a^2+x^2)}\tan\theta$. $\therefore$ $dy=\sqrt{(a^2+x^2)}\sec^2\theta\,d\theta$.

When $y=0$, $\theta=0$ and $y=a$, then $\tan\theta=\dfrac{a}{\sqrt{(a^2+x^2)}}$, so that

$$\sin\theta=\frac{a}{\sqrt{(2a^2+x^2)}} \qquad \ldots(1)$$

and hence the limits of θ are 0 to θ when $\tan\theta$ or $\sin\theta$ is as written above.

$$\therefore \quad V=\int_0^a\int_0^\theta\frac{a^4\,\sqrt{(a^2+x^2)}\sec^2\theta\,dx\,d\theta}{(a^2+x^2)^{3/2}\sec^3\theta}$$

$$=a^4\int_0^a\int_0^\theta\frac{1}{(a^2+x^2)}\cos\theta.dx\,d\theta$$

$$=a^4\int_0^a\frac{1}{a^2+x^2}\Big[\sin\theta\Big]_0^\theta dx=a^4\int_0^a\frac{1}{a^2+x^2}\sin\theta\,dx,$$

Now put $\sin=\dfrac{a}{\sqrt{(2a^2+x^2)}}$ by (1).

$$\therefore \quad V=a^4\int_0^a\frac{a}{(a^2+x^2)\sqrt{(2a^2+x^2)}}\,dx.$$

Put $x=a\sqrt{2}\tan\phi$, $\therefore\ dx=a\sqrt{2}\sec^2\phi\, d\phi$.

Also when $x=0$, $\phi=0$; when $x=a$, $\tan\phi=\dfrac{1}{\sqrt{2}}$.

$$\therefore\ \sin\phi=\frac{1}{\sqrt{3}}. \qquad \ldots(2)$$

$$\therefore\ V=a^5\int_0^{\phi}\frac{a\sqrt{2}\sec^2\phi\, d\phi}{a^2(1+2\tan^2\phi)\, a\sqrt{2}\sec\phi}$$

$$=a^3\int_0^{\phi}\frac{\cos\phi\, d\phi}{\cos^2\phi+2\sin^2\phi}$$

$$=a^3\int_0^{\phi}\frac{\cos\phi\, d\phi}{1+\sin^2\phi}=a^3\left[\tan^{-1}(\sin\phi)\right]_0^{\phi}$$

$$=a^3[\tan^{-1}\sin\phi]=a^3\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\ [\text{by } (2)]$$

$$=a^3.\pi/6=\pi a^3/6$$

Ex. 36. *Find the volume cut off from the paraboloid $x^2+y^2=4az$ by the plane $x+y+z=a$.*

Here z varies from $(a-x-y)$ to $\dfrac{1}{4a}(x^2+y^2)$.

Again in order to find the limits of y we eliminate z between the given equations and we get $x^2+y^2=4a(a-x-y)$

or $$y^2+4ay+(x^2+4ax-4a^2)=0. \qquad \ldots(1)$$

If the limits of y be from y_1 to y_2, then y_1 and y_2 are the roots of above.

$$\therefore\ y_1+y_2=-4a \text{ and } y_1y_2=x^2+4ax-4a^2=(x+2a)^2-8a^2. \qquad \ldots(2)$$

$$V=\iiint dx\, dy\, dz=\iint\Big[z\Big]_{a-x-y}^{(x^2+y^2)/4a}dx\, dy$$

$$=\iint\left[\frac{x^2+y^2}{4a}-(a-x-y)\right]dx\, dy$$

$$=\frac{1}{4a}\iint_{y_1}^{y_2}[(x^2+4ax-4a^2)+4ay+y^2]\, dx\, dy$$

$$=\frac{1}{4a}\iint\left[(x^2+4ax-4a^2)\,y+2ay^2+\frac{y^3}{3}\right]_{y_1}^{y_2}dx$$

$$=\frac{1}{4a}\int(y_2-y_1)\,[(x^2+4ax-4a^2)+2a\,(y_1+y_2)$$

$$+\tfrac{1}{3}\,(y_2^2+y_1y_2+y_1^2)]\,dx.$$

Now $y_1+y_2=-4a$ and also $y_1y_2=x^2+4ax-4a^2$ by (2).

$$\therefore\quad V=\frac{1}{4a}\int(y_2-y_1)\left[y_1y_2-\frac{(y_1+y_2)^2}{2}+\tfrac{1}{3}\,\{(y_1+y_2)^2-y_1y_2\}\right]dx$$

$$=\frac{1}{4a}\int(y_2-y_1)\left[\frac{4y_1y_2-(y_1+y_2)^2}{6}\right]dx$$

$$=\frac{1}{24a}\int(y_1-y_2)\,(y_1-y_2)^2\,dx$$

$$=\frac{1}{24a}\int\left[(y_1+y_2)^2-4y_1y_2\right]^{3/2}dx$$

$$=\frac{1}{24a}\int[16a^2-4\,\{(x+2a)^2-8a^2\}]^{3/2}\,dx$$

$$=\frac{4\sqrt{4}}{24a}\int\{12a^2-(x+2a)^2\}^{3/2}\,dx$$

$$=\frac{1}{3a}\int_{-k}^{k}(k^2-t^2)^{3/2}\,dx\text{ where }k^2=12a^2\text{ and }t^2=(x+2a)^2$$

Put $t=k\sin\theta$.

$$=\frac{1}{3a}\int_{-\pi/2}^{\pi/2}k^3\cos^3\theta.k\cos\theta\,d\theta=\frac{2}{3a}\,k^4\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=\frac{2}{3a}\,(12a^2)^2.\frac{3.1}{4.2}.\frac{\pi}{2}=18\pi a^3.$$

Ex. 36. *Find the volume of the portion of the paraboloid* $\frac{x^2}{a}+\frac{y^2}{b}=2z$ *cut off by the plane* $lx+my+nz=p$. **(Agra 60)**

It is exactly as Q 35

z varies from $\frac{p-lx-my}{n}$ to $\frac{1}{2}\left(\frac{x^2}{a}+\frac{y^2}{b}\right)$.

Again in order to find the limits of y we eliminate z and we get

$$\frac{x^2}{a}+\frac{y^2}{b}=2\frac{(p-lx-my)}{n}$$

or

$$\frac{y^2}{b}+\frac{2m}{n}y+\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right)=0.$$

If the limits of y be from y_1 to y_2, then y_2 and y_1 are the roots of above.

$$\therefore \quad y_1+y_2=-\frac{2mb}{n} \text{ and } y_1y_2=b\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right). \quad \ldots(1)$$

$$\therefore \quad V=\iiint dx\,dy\,dz=\iint\Big[z\Big]_{(p-lx-my)/n}^{\frac{1}{2}(x^2/a+y^2/b)} dx\,dy$$

$$=\iint\left[\tfrac{1}{2}\left(\frac{x^2}{a}+\frac{y^2}{b}\right)-\left(\frac{p-lx-my}{n}\right)\right]dx\,dy$$

$$=\tfrac{1}{2}\iint\left[\left(\frac{x^2}{a}+\frac{2lx}{n}-\frac{2p}{n}\right)y+\frac{2m}{n}\cdot\frac{y^2}{2}+\frac{y^3}{3b}\right]_{y_1}^{y_2}dx$$

$$=\tfrac{1}{2}\int(y_2-y_1)\left[\frac{y_1y_2}{b}-\frac{m}{n}(y_2+y_1)+\frac{1}{3b}(y_2^2+y_1^2+y_1y_2)\right]dx$$

[by (1)]

$$=\tfrac{1}{2}\int(y_2-y_1)\left[\frac{y_1y_2}{b}-\frac{(y_2+y_1)^2}{2b}+\frac{1}{3b}(y_2^2+y_1^2+y_1y_2)\right]dx$$

[by (1)]

$$=\tfrac{1}{2}\int\frac{(y_2-y_1)}{6b}\,[6y_1y_2-3(y_2+y_1)^2+2\{(y_1+y_2)-y_1y_2\}]\,dx$$

$$=\frac{1}{12b}\int(y_2-y_1)\,[4y_1y_2-(y_1+y_2)^2]\,dx$$

$$=\frac{1}{12b}\int(y_1-y_2)\,[(y_1+y_2)^2-4y_1y_2]\,dx.$$

$$V=\frac{1}{12b}\int[(y_1-y_2)^2]^{3/2}\,dx. \quad \ldots(2)$$

Now $(y_1-y_2)^2=(y_1+y_2)^2-4y_1y_2=\dfrac{4m^2b^2}{n^2}-4b\left(\dfrac{x^2}{a}+\dfrac{2lx}{n}-\dfrac{2p}{n}\right)$

$$=\frac{4b}{a}\left[-x^2-\frac{2la}{n}x+\frac{2pa}{n}+\frac{m^2ba}{n^2}\right]$$

$$=\frac{4b}{a}\left[-\left(x+\frac{la}{n}\right)^2+\frac{l^2a^2}{n^2}+\frac{2pa}{n}+\frac{m^2ba}{n^2}\right]$$

$$=\frac{4b}{a}\left[\frac{a}{n^2}(al^2+2pn+bm^2)-\left(x+\frac{la}{n}\right)^2\right]$$

$$=\frac{4b}{a}[k^2-t^2]$$

where $k^2=\frac{a}{n^2}(al^2+2pn+bm^2)$ and $t^2=\left(x+\frac{la}{n}\right)^2$ ...(2)

$$\therefore \quad V=\frac{1}{12b}\int_{-k}^{k}\left[\frac{4b}{a}(k^2-t^2)\right]^{3/2}dt$$

$$=\frac{1}{12b}\frac{4b}{a}\sqrt{\left(\frac{4b}{a}\right)}.2\int_0^k(k^2-t^2)^{3/2}dt. \qquad \text{Put } t=k\sin\theta$$

$$=\frac{4}{3}\frac{\sqrt{b}}{a\sqrt{a}}\int_0^{\pi/2}k^3\cos^3\theta.k\cos\theta\,d\theta$$

$$=\frac{4}{3}\frac{\sqrt{b}}{a\sqrt{a}}\frac{a^2}{n^4}(al^2+2pn+bm^2)^2.\left[\frac{3.1}{4.2}.\frac{\pi}{2}\right]$$

$$=\frac{\pi}{4}\frac{\sqrt{(ab)}}{n^4}(al^2+2pn+bm^2)^2.$$

Ex. 37. *Prove that the volume cut off from the paraboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$ *by the plane* $z=px+qy+r$ *is*

$$\frac{\pi abc}{4}\left(\frac{a^2p^2}{c^2}+\frac{b^2q^2}{c^2}+\frac{2r}{c}\right)^2.$$

Proceed exactly as in Q. 35 and 36.

§ 2. Volume in polar co-ordinates.

We know that in cartesian co-ordinates $V=\int dx\,dy\,dz$.

As we shall show in the next chapter by substituting

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$$

such that $x^2+y^2=r^2\sin^2\theta$ and $x^2+y^2+z^2=r^2$.

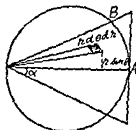
Corresponding formula in polar coordinates becomes

$$V=\iiint r^2 \sin\theta \, d\phi \, d\theta \, dr.$$

Ex. 38. *A right cone has its vertex in the surface of the sphere and its axis coincident with the diameter of the sphere passing through that point. Find the volume common to the cone and the sphere.* **(Agra 55)**

1st method by double integration.

It is clear from the figure that the volume common to the cone and the sphere is obtained by revolution of the area OAB about the axis OA, the axis of the cone whose semi-vertical angle is say α.



Now consider an elementary area $r\,d\theta\,dr$.

If we revolve the area about OA, it will generate a ring whose radius as shown in the figure is $r\sin\theta$ and hence the volume of this elementary ring will be

$$2\pi\,(r\sin\theta)\,r\,d\theta\,dr.$$

Hence the total volume is $\iint 2\pi\,(r\sin\theta)\,r\,d\theta\,dr$. Clearly the equation of the circle on OA as diameter is $r=2a\cos\theta$.

$$\therefore \quad V=\int_0^{\alpha}\int_0^{2a\cos\theta} 2\pi\sin\theta . r^2\,d\theta\,dr$$

$$=2\pi\int_0^{\alpha}\sin\theta . \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$=\frac{16\pi a^3}{3}\int_0^{\alpha}\cos^3\theta\sin\theta\,d\theta$$

$$=\frac{16\pi a^3}{3}\left[-\frac{\cos^4\theta}{4}\right]_0^{\alpha}=\frac{4\pi a^3}{3}\,[1-\cos^4\alpha]$$

2nd method by triple integration.

Let us choose z-axis along OA and the radius of the sphere a so that its centre is (0, 0, 0) and hence its cartesian equation is $(x-0)^2+(y-0)^2+(z-a)^2=a^2$,

or $$x^2+y^2+z^2=2az.$$

Converting into polar co-ordinates, we get

$$r^2=2ar\cos\theta \quad \text{or} \quad r=2a\cos\theta,$$

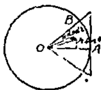
$$\therefore \quad V=\int_0^{2\pi}\int_0^{\alpha}\int_0^{2a\cos\theta} r^2\sin\theta\, d\phi\, d\theta\, dr$$

$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\alpha}(2a\cos\theta)^3\sin\theta\, d\phi\, d\theta$$

$$=\frac{8a^3}{3}\int_0^{2\pi}\left[-\frac{\cos^4\theta}{4}\right]_0^{\alpha} d\phi=\frac{2a^3}{3}\int_0^{2\pi}(1-\cos^4\alpha)\, d\phi$$

$$=\frac{2a^3}{3}(1-\cos^4\alpha)\left[\phi\right]_0^{2\pi}=\frac{4\pi a^3}{3}(1-\cos^4\alpha).$$

Ex. 39. *A right cone has its vertex at the centre of a sphere and its axis coincident with the diameter of the sphere passing through that point. Find the volume common to the cone and sphere.*



1st Method. Here the equation of the circle with centre at O is $r=a$ and the semi-vertical angle of the cone is α. Arguing as in Q. 37, we get

$$V=\int_0^{\alpha}\int_0^{a} 2\pi(r\sin\theta)\, r\, d\theta\, dr$$

$$=2\pi\int_0^{\alpha}\sin\theta.\left[\frac{r^3}{3}\right]_0^{a} d\theta=\frac{2\pi a^3}{3}\left[-\cos\alpha\right]_0^{\alpha}$$

$$=\frac{2\pi a^3}{3}(1-\cos\alpha).$$

2nd Method. $V=\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^2\sin\theta\, d\phi\, d\theta\, dr$

or $$V=\int_0^{2\pi}\int_0^{\alpha}\frac{a^3}{3}\sin\theta\,d\phi\,d\theta=\frac{a^3}{3}\int_0^{2\pi}\Big[-\cos\theta\Big]_0^{\alpha}d\phi$$
$$=\frac{a^3}{3}(1-\cos\alpha)\int_0^{2\pi}d\phi=\frac{a^3}{3}(1-\cos\alpha)\Big[\phi\Big]_0^{2\pi}$$
$$=\frac{2\pi a^3}{3}(1-\cos\alpha).$$

Ex. 40. *Find the whole volume of the solid bounded by the surface* $(x^2+y^2+z^2)^3=27a^3xyz$ **(Agra 40)**

Putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, we get
$$(r^2)^3=27a^3.r^3.\sin^2\theta\cos\theta\sin\phi\cos\phi.$$
$$\therefore\quad r=3a(\sin^2\theta\cos\theta\sin\phi\cos\phi)^{1/3}.$$
$$\therefore\quad V=4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3a(\sin^2\theta\cos\theta\sin\phi\cos\phi)^{1/3}}r^2.\sin\theta\,d\phi\,d\theta\,dr$$
$$=4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^3}{3}\right]\sin\theta\,d\phi\,d\theta$$
$$=4\int_0^{\pi/2}\int_0^{\pi/2}\frac{27a^3\sin^2\theta\cos\theta\sin\phi\cos\phi}{3}\sin\theta\,d\phi\,d\theta$$
$$=36a^3\int_0^{\pi/2}\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2}\sin\phi\cos\phi\,d\phi$$
$$=9a^3\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}=\frac{9}{2}a^3.$$

§ 3. Area of surface.

Let the equation of the surface be $z=f(x,y)$ and (x,y,z) be the coordinates of any point on it. Let the elementary surface enclosing the point (x,y,z) be denoted by δs. Now the projection of δs on the plane $z=0$ is $\delta x.\delta y$.

Also from co-ordinate geometry, we know that
$$A_s=A\cos\theta,$$
where A is the area and A_s the projection of area on the plane $z=0$ and θ is the angle between the plane of the area and plane of projection.

Hence $\delta x\,\delta y$ — projection of δS on $z=0=\delta S\cos\gamma$.

or $$\delta S=\delta x\ \delta y \sec\gamma, \qquad \ldots(1)$$
where γ is the angle between the plane $z=0$ and the tangent plane to the given surface at (x, y, z). Also we know from Co-ordinate Geometry that

$$\sec\gamma=\sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}. \qquad \ldots(2)$$

Hence from (1) by the help of (2),

$$\delta S=\delta x\ \delta y.\sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}.$$

If S be the total area of the surface, then

$$\mathbf{S}=\iint\sqrt{\left\{1+\left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}}\right)^2+\left(\frac{\partial \mathbf{z}}{\partial \mathbf{y}}\right)^2\right\}}\ \mathbf{dx\ dy}.$$

The limits of integration for x and y are taken so as to cover the whole region of projection of S on $z=0$.

Polar co-ordinates.

Changing to polar co-ordinates the above formula for surface takes the form

$$\mathbf{S}=\iint\left\{\mathbf{r}^2\left(\frac{\partial \mathbf{r}}{\partial\phi}\right)^2+\mathbf{r}^2\sin^2\theta\left(\frac{\partial \mathbf{r}}{\partial\theta}\right)^2+\mathbf{r}^4\sin^2\theta\right\}^{1/2}\mathbf{d}\theta\ \mathbf{d}\phi.$$

Note. We shall be taking

$$S=\iint\sec\gamma\ dx\ dy,$$

where $$\sec\gamma=\sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}.$$

Ex. 41. *Show that the area of the surface $z^2=2xy$ included between the plane $x=0$, $x=a$, $y=0$, $y=b$ is*

$$4\sqrt{(ab)}.\frac{(a+b)}{3\sqrt{2}}. \qquad \textbf{(Agra 63)}$$

From the relation $z^2=2xy$, we get $2z\frac{\partial z}{\partial x}=2y$.

$$\therefore\ \frac{\partial z}{\partial x}=\frac{y}{z} \text{ and similarly } \frac{\partial z}{\partial y}=\frac{x}{z}.$$

$$\therefore \quad \sec\gamma = \sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}} = \sqrt{\left(1+\frac{x^2+y^2}{z^2}\right)}$$

$$= \frac{\sqrt{(x^2+y^2+z^2)}}{z}$$

$$= \frac{\sqrt{(x^2+y^2+2xy)}}{\sqrt{(2xy)}} = \frac{1}{\sqrt{2}}\left[\frac{x+y}{\sqrt{(xy)}}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{\left(\frac{x}{y}\right)}+\sqrt{\left(\frac{y}{x}\right)}\right].$$

$$\therefore \quad S = \int_0^a \int_0^b \sec\gamma \, dx \, dy$$

$$= \frac{1}{\sqrt{2}} \int_0^a \int_0^b \left[\sqrt{\left(\frac{x}{y}\right)}+\sqrt{\left(\frac{y}{x}\right)}\right] dx \, dy$$

$$= \frac{1}{\sqrt{2}} \int_0^a \left[\sqrt{x} \, . \, 2\sqrt{y} + \frac{1}{\sqrt{x}} \cdot \tfrac{2}{3} y^{3/2}\right]_0^b$$

$$= \frac{1}{\sqrt{2}} \int_0^a \left\{2\sqrt{b}\sqrt{x} + \tfrac{2}{3} b\sqrt{b} \, \frac{1}{\sqrt{x}}\right\} dx$$

$$= \frac{1}{\sqrt{2}} \left[2\sqrt{b} \, . \, \tfrac{2}{3} x^{3/2} + \tfrac{2}{3} b\sqrt{b} \, . \, 2\sqrt{x}\right]_0^a$$

$$= \frac{4}{3\sqrt{2}} \left[\sqrt{b} \, . \, a\sqrt{a} + b\sqrt{b} \, \sqrt{a}\right]$$

$$= \frac{4}{3\sqrt{2}} \sqrt{(ab)} \, (a+b).$$

Ex. 42. *Find the area of the surface of the sphere $x^2+y^2+z^2=a^2$ included within the cylinder $x^2+y^2=ax$.*

(Agra 49)

Clearly $\frac{\partial z}{\partial x} = -\frac{x}{z}$ and $\frac{\partial z}{\partial y} = -\frac{y}{z}$.

$$\therefore \quad \sec\gamma = \sqrt{\left\{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}$$

$$= \frac{a}{z} = \frac{a}{\sqrt{(a^2-x^2-y^2)}}$$

Again changing to polars by putting

$$x = r\cos\theta,\ y = r\sin\theta,$$

$$\sec\gamma = \frac{a}{\sqrt{(a^2-r^2)}}.$$

Also $x^2+y^2=ax$ changes to $r^2 = ar\cos\theta$

or $\quad r = a\cos\theta$ and $dx\,dy = r\,d\theta\,dr$

$$\therefore\quad S = \iint \sec\gamma\,dx\,dy = 4\int_0^{\pi/2}\int_0^{a\cos\theta} \frac{a}{\sqrt{(a^2-r^2)}}\,r\,d\theta\,dr$$

$$= 4a\int_0^{\pi/2}\Big[-\sqrt{(a^2-r^2)}\Big]_0^{a\cos\theta} d\theta$$

$$= 4a\int_0^{\pi/2}(a - a\sin\theta)\,d\theta$$

$$= 4a^2\Big[\theta+\cos\theta\Big]_0^{\pi/2} = 4a^2\left[\frac{\pi}{2}-1\right] = 2a^2(\pi-2).$$

Ex. 43. *Find the area of the surface of the sphere* $x^2+y^2+z^2=a^2$, *which lies inside the cylinder* $x^2+y^2=ay$.

Proceeding exactly as above, $x^2+y^2=ay$ transform to

$$r = a\sin\theta.$$

$$\therefore\quad S = 4\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{a}{\sqrt{(a^2-r^2)}}\,r\,d\theta\,dr = \text{etc.} = 2a^2(\pi-2).$$

Ex. 44. *Show that the area of the surface of the sphere* $x^2+y^2+z^2=a^2$ *which lies inside the cylinder* $x^2+y^2+z^2=ay$ *is* $2a^2(\pi-2)$.

Proceeding as in Q. 42 and 43,

$$\sec\gamma = \frac{a}{\sqrt{(a^2-x^2-y^2)}} = \frac{a}{\sqrt{(a^2-r^2)}}.$$

Also projection of $x^2+y^2+z^2=ay$ on the plane $z=0$ is $x^2+y^2=ay$ which is same cylinder as in Q. 43.

Ex. 45. *Find the surface of* $x^2+z^2=a^2$ *that lies inside the cylinder* $x^2+y^2=a^2$. **(Vikram 65)**

$$\sec\gamma = \sqrt{\left(1+\frac{x^2}{z^2}\right)} = \frac{a}{z} = \frac{a}{\sqrt{(a^2-x^2)}}.$$

$$\therefore \quad S=8\int_0^a \int_0^{\sqrt{(a^2-x^2)}} \frac{a}{\sqrt{(a^2-x^2)}}\,dx\,dy$$

$$=8\int_0^a \left[\frac{a}{\sqrt{(a^2-x^2)}}\cdot y\right]_0^{\sqrt{(a^2-x^2)}} dx$$

$$=8\int_0^a a\,dx=8a^2$$

Ex. 46. *Find the area of the surface* $x^2+y^2+z^2=a^2$ *inside the surface* $(x^2+y^2)^2=a^2\,(x^2-a^2)$

As in Q. 42, $\sec\gamma=\dfrac{a}{\sqrt{(a^2-r^2)}}$, where $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy=r\,d\theta\,dr$. Also $(x^2+y^2)=a^2\,(x^2-y^2)$ transforms to $(r^2)^2=a^2r^2\,(\cos^2\theta-\sin^2\theta)$ or $r^2=a^2\cos 2\theta$, which is the equation of the well-known curve Lemniscate of Bernoulli (fig. P. 138) It consists of two loops and one loop is formed between $\theta=-\pi/4$ to $\theta=\pi/4$

$$\therefore \quad S=8\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \frac{a}{\sqrt{(a^2-r^2)}}\,r\,d\theta\,dr$$

$$=8a\int_0^{\pi/4}\left[-\sqrt{(a^2-r^2)}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$=8a\int_0^{\pi/4} -\{a\sqrt{(1-\cos 2\theta)}-a\}\,d\theta$$

$$=8a^2\int_0^{\pi/4}(1-\sqrt{2}\sin\theta)\,d\theta$$

$$=8a^2\left[\theta+\sqrt{2}\cos\theta\right]_0^{\pi/4}$$

$$=8a^2\left[\frac{\pi}{4}+1-\sqrt{2}\right].$$

Ex. 47. *Find the area of*

inside the cylinder $\ldots^2+y^2)^2=2a^2x$

Here

$$=\frac{\sqrt{(a^2+r^2)}}{a}, \text{ where } x=r\cos\theta,\ y=r\sin\theta.$$

Also the cylinder $(x^2+y^2)^2=2a^2xy$ changes to

$$(r^2)^2=2a^2.r^2\sin\theta\cos\theta \quad \text{or} \quad r^2=a^2\sin 2\theta.$$

It also consists of two loops and one loop is formed between $\theta=0$ and $\theta=\pi/2$.

$$S=\iint \sec\gamma\, dx\, dy=2\int_0^{\pi/2}\int_0^{a\sqrt{(\sin 2\theta)}}\frac{\sqrt{(a^2+r^2)}}{a}.r\, d\theta\, dr$$

$$=\frac{2}{a}\int_0^{\pi/2}\frac{1}{2}.\frac{2}{3}\left[(a^2+r^2)^{3/2}\right]_0^{a\sqrt{(\sin 2\theta)}}d\theta$$

$$=\frac{2}{3a}\int_0^{\pi/2}a^3\{(1+\sin 2\theta)^{3/2}-1\}\,d\theta.$$

Now $1+\sin 2\theta=\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta$

$$=(\cos\theta+\sin\theta)^2.$$

$\therefore\quad (1+\sin 2\theta)^{3/2}=(\cos\theta+\sin\theta)^3$

$$=\cos^3\theta+\sin^3\theta+3\cos^2\theta\sin\theta+3\cos\theta\sin^2\theta.$$

$$\therefore\quad S=\tfrac{2}{3}a^2\int_0^{\pi/2}\{\cos^3\theta+\sin^3\theta+3\cos^2\theta\sin\theta+3\sin^2\theta\cos\theta-1\}\,d\theta$$

$$=\frac{2}{3}a^2\left[\frac{2}{3}+\frac{2}{3}+3.\frac{1}{3}+3.\frac{1}{3}-\frac{\pi}{2}\right]$$

$$=\frac{2}{3}a^2\left[\frac{10}{3}-\frac{\pi}{2}\right]=\frac{2}{3}a^2\left[\frac{20-3\pi}{6}\right]=\frac{a^2}{9}(20-3\pi).$$

Ex. 48. *Show that the area of the surface of the paraboloid $az=x^2+y^2$ which lies between the planes $z=0$ and $z=a$ is $\frac{\pi}{6}(5\sqrt{5}-1)\,a^2$*

$$\frac{\partial z}{\partial x}=\frac{2x}{a} \text{ and } \frac{\partial z}{\partial y}=\frac{2y}{a};\quad \therefore\quad \sec\gamma=\sqrt{\left[1+\frac{4}{a^2}(x^2+y^2)\right]}$$

$$=\frac{\sqrt{(a^2+4r^2)}}{a}.$$

Projection of the surface on the plane $z=0$ is $x^2+y^2=a^2$ or $r=a$ which is a circle between $\theta=0$ and $\theta=2\pi$.

$$\therefore \quad S=\iint \sec \gamma . dx\, dy=\int_0^{2\pi} \int_0^a \frac{\sqrt{(a^2+4r^2}}{a}\, r\, d\theta\, dr$$

$$=\frac{1}{8a} \int_0^{2\pi} \int_0^a \sqrt{(a^2+4r^2)}\; 8r\, d\theta\, dr$$

$$=\frac{1}{8a} \int_0^{2\pi} \left[\frac{2}{3}\,(a^2+4r^2)^{3/2}\right]_0^a d\theta$$

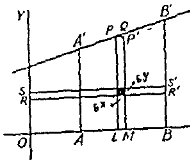
$$=\frac{1}{12a} \int_0^{2\pi} (5\sqrt{5}-1)\, a^3\, d\theta$$

$$=\frac{a^2}{12}(5\sqrt{5}-1)\Big[\theta\Big]_0^{2\pi}=\frac{\pi}{6}\,(5\sqrt{5}-1)\, a^2.$$

CHAPTER V

MULTIPLE INTEGRALS

§ 1. **Double integration (Cartesian).** Students are conversant with the formula $\int_a^b y\,dx$ which stands for area between any curve $y=f(x)$, x-axis and ordinates at $x=a$ and $x=b$. It will be very much convenient to use double integration in calculation of area as explained below :—



Let A', B' be any two points on the curve $y=f(x)$ $[a, f(a)]$ and $[b, f(b)]$ respectively, and we are required to find the area $A'B'BA$. Again choose two neighbouring points P and Q respectively (x, y) and $(x+\delta x, y+\delta y)$. From these points draw PL and QM parallel to y-axis, so that $LM=\delta x$. Again draw RR' and SS' parallel to axis of x at distances y and $y+\delta y$ from the origin, so that $RS=\delta y$. From the figure it is clear that the area enclosed between these four lines PL, QM, RR' and SS' is clearly $\delta x\,\delta y$. Hence the area

$$PP'\,LM=\underset{\delta y\to 0}{\text{Lt}}\ \Sigma\,\delta x\,\delta y,$$

or the area of the elementary strip is $\left[\int dy\right]\delta x$ where the limits of integration for y are $y=0$ are $y=f(x)$.

Hence elementary strip is $\left[\int_0^{f(x)} dy\right] \delta x$.

It may however be noted that in performing this integration x is to be regarded as constant.

Again the total area $A'B'BA$ is the sum of all the areas of these types of elementary parallel strips and hence it is equal to $\underset{\delta x \to 0}{\text{Lt}} \Sigma \left[\int_0^{f(x)} dy\right] \delta x = \int \left[\int_0^{f(x)} dy\right] dx$.

The limits of integration for x are from $x=a$ to $x=b$.

$\therefore$ Area $A'B'BA = \int_a^b \left[\int_0^{f(x)} dy\right] dx$

The above formula is generally written as $\int_a^b \int_0^{f(x)} dx\, dy$.

The first integration is to be performed with respect to right hand variable i.e. y whose limits of integration are functions of x and then the integration w.r.t. x is performed whose limits are constant.

Note :—In case the area be enclosed between the curves $y=f(x)$ and $y=\phi(x)$, then the corresponding formula will be

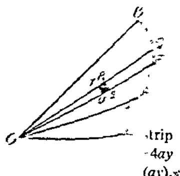
$$\int_a^b \int_{f(x)}^{\phi(x)} dx\, dy.$$

Here firstly we added all the elements in a strip parallel to y-axis and then found in which x was constant and then we found the total area as summation of all such types of strips which are parallel to y-axis.

If, however, we change the order of integration . $\iint dy\, dx$ instead of $\iint dx\, dy$, then
be done w.r.t. x treating y as

add all the elements in a strip parallel to x-axis (y constant) and then sum of all such strips parallel to x-axis will give the total area This is known as change of order of integration. In this case we will have to find the limits of x as function of y (as the integration is firstly done w.r.t. x) and then the limits of y as constants. In order to find the new set of limits we should draw a clean figure and then divide the total area into different parts by drawing lines parallel to x-axis at the points where the elementary strips drawn parallel to x-axis change their character, *i.e.* when the extremities of each strip lie on the same two curves, it will form one region and the point from where they lie on another set of two curves, it will form another region. We find the area of each such part and then add. This working rule will be clear from the following examples.

§ 2. Double integration : Polar Co-ordinates.



Let the equation of the curve be $r=f(\theta)$ and the vectorial angles of points A and B be $\theta=\alpha$ and $\theta=\beta$ respectively. Let the co-ordinates of two neighbouring points P and Q be (r, θ) and $(r+\delta r, \theta+\delta\theta)$. With centre O draw two circular arcs TU and RS of radii r and $r+\delta r$ respectively so that the elementary area $TRSU$ is

$$=\underset{\delta\theta\to 0}{\text{Lt}}\ \Sigma\left[\int_0^{f(\theta)} r\,dr\right]\delta\theta$$

$$=\int_\alpha^\beta\left[\int_0^{f(\theta)} r\,dr\right]d\theta.$$

Above is written as $\int_\alpha^\beta \int_0^{f(r)} \mathbf{r}\,d\theta\,\mathbf{dr}$.

The first integration is to be performed w.r.t. the right hand variable r and while performing this integration, θ is to be regarded as constant.

Note. *In double integration if the limits of both variables are constant, then you can always change the order of integration. But in case the limits of y are functions of x, then while changing the order of integration (i.e. performing first integration w.r t. x) you will have to find the new limits of x as functions of y by consideration of geometrical conditions as will be clear from the following examples.*

The general form of double integral is

$$\iint V\,dx\,dy,$$

where V is a function of x, y.

Exercise

Ex. 1. *Change the order of integration in*

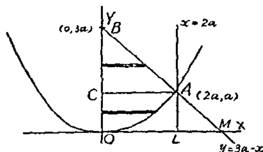
$$\int_0^{2a}\int_{x^2/4a}^{3a-x} F(x, y)\,dx\,dy.$$

(Rajasthan 48, ; Agra 48, 54)

The limits of integration give the region of integration to be bounded by

$$y=\frac{x^2}{4a},\ y=3a-x \text{ and } x=0,\ x=2a,$$

i.e. parabola $x^2=4ay$ and line $y=3a-x$ and ordinates $x=0$ and $x=2a$.



First of all, we draw the figure. The line $y=3a-x$ meets the parabola $x^2=4ay$, where

$x^2=4a\ (3a-x)$

or $x^2+4ax-12a^2=0$

or $(x+6a)\ (x-2a)=0$

i.e. $x=-6a$ and $2a$.

When $x=2a$, $y=a$ Thus the point of intersection is $(2a, a)$ We do not consider the other value of x, which is —ive as the limits of x are from 0 to $2a$. Also putting $x=0$ in the equation of line, the point B is $(0, 3a)$.

Thus from the figure the region of integration is OAB. The given integral is when we have considered strips parallel to y-axis and we have to change its order so that we have now to consider the strips parallel to x-axis. Through A draw a line AC parallel to x-axis, thus dividing the region into two separate parts OAC and ACB. In each part take elementary strip parallel to x-axis. In the given integration we were to integrate with respect to y first whose limits were given as functions of x. Now after change of order we have to integrate w.r.t. x first whose limits as functions of y have to be determined.

Region OAC. In this region the elementary strip parallel to x-axis has its extremities on $x=0$ and $x^2=4ay$ and hence the limits of integration for x are 0 to $2\sqrt{(ay)}$. Again the constant lengths of y are clearly $y=0$ for O and $y=a$ for A.

$\therefore$ After change of order the integration for the region $OAC=\int_0^a \int_0^{2\sqrt{(ay)}} F\ (x, y)\ dy\ dx$.

Region CAB. In this region the elementary strip parallel to x-axis has its extremities on $x=0$ and $y=3a-x$ and hence the limits of integration for x are 0 to $3a-y$. Again the constant limits of y are clearly $y=a$ for A and $y=3a$ for B

$\therefore$ after change of order, the integration for the region $CAB=\int_0^{3a}\int_0^{3a-x} F(x, y)\, dy\, dx$ ...(2)

$$\therefore \quad \int_0^{2a}\int_{x^2/4a}^{3a-x} F(x, y)\, dx\, dy \text{ i.e. given integral}$$

$$=\int_0^a \int_0^{2\sqrt{(ay)}} F(x, y)\, dy\, dx+\int_0^{3a}\int_0^{3a-y} F(x, y)\, dy\, dx.$$

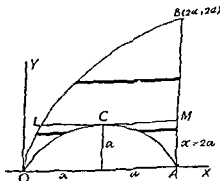
Ex. 2. *Change the order of integration in the double integral* $\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V\, dx\, dy$.

(Sagar 63 ; Vikram 64 ; Rajputana 49, 58 ; Agra 50, 57, 60, 62)

The limits of integration give the region of integration to be bounded by

$y=\sqrt{(2ax-x^2)}$ *i.e.* $x^2+y^2=2ax$, *i.e.* $(x-a)^2+y^2=a^2$

i.e. a circle whose centre is $(a, 0)$ and radius a and $y=\sqrt{(2ax)}$, *i.e.* parabola $y^2=2ax$ and $x=0$ and $x=2a$. The parabola and circle touch each other at $(0, 0)$, the common tangent thereat being $x=0$. The line $x=2a$ meets the parabola $y^2=2ax$ at $y=2a$ *i.e.* point B it $(2a, 2a)$: Thus the region of integration is given by $OBACO$. Through C draw a line parallel to x-axis. Now we shall



divide the region into three parts in each of which the extremities of strips drawn parallel to axis of x lie on different sets of curves namely region OLC, AMC and LMB.

Region OLC. The extremities of the strip parallel to axis of x lie on $y^2=2ax$ and $x^2+y^2=2ax$, so that the limits of integration of x are $y^2/2a$ to $a-\sqrt{(a^2-y^2)}$ because from $x^2-2ax+y^2=0$, we have $x=a\pm\sqrt{(a^2-y^2)}$ and we have chosen $-$ sign from $\pm$ because x is less than or equal to a for the region. Also the constant limits of y are $y=0$ for O and $y=a$ for C.

Hence after change of order the integration for this region OLC is
$$\int_0^a \int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} V\, dy\, dx. \qquad \ldots(1)$$

Region AMC. The extremities of strips parallel to axis of x lie on circle and line $x=2a$ and also x is greater than a and hence the limits of x are $a+\sqrt{(a^2-y^2)}$ to $2a$ and the limits of y are $y=0$ for A and $y=a$ for C.

Hence after change of order the integration for the region AMC is
$$\int_0^a \int_{a+\sqrt{(a^2-y^2)}}^{2a} V\, dy\, dx. \qquad \ldots(2)$$

Region LMB. The extremities of strips parallel to axis of x lie on the parabola $y^2=2ax$ and the line $x=2a$ and hence the limits of x are $\frac{y^2}{2a}$ to $2a$ and the limits of y are $y=a$ for L and $y=2a$ for B.

Hence after change of order the integration for the region LMB is
$$\int_a^{2a} \int_{y^2/2a}^{2a} V\, dy\, dx. \qquad \ldots(3)$$

Hence the given double integral

$=$ sum of the integrals no. (1), (2) and (3).

Ex. 3. *Change the order of integration in the double integral* $\int_0^a \int_{\sqrt{(ax-x^2)}}^{\sqrt{(ax)}} f(x, y)\, dx\, dy.$ (Agra 60)

It is exactly same question as Ex 2

Circle is

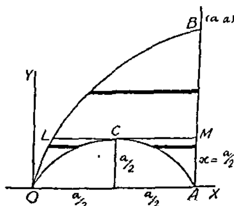
$x^2+y^2-ax=0$

or $\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2.$

Also

$y=\frac{a\pm\sqrt{(a^2-4y^2)}}{2}.$

Also parabola meets $x=a$ at (a, a)



Hence arguing as above in Ex. 2 the given integral

$$=\int_0^{a/2}\int_{y^2/a}^{\frac{1}{2}[a-\sqrt{(a^2-4y^2)}]} f(x, y)\, dy\, dx$$

for OLC

$$+\int_0^{a/2}\int_{\frac{1}{2}[a+\sqrt{(a^2-4y^2)}]}^{a} f(x, y)\, dy\, dx$$

for AMC

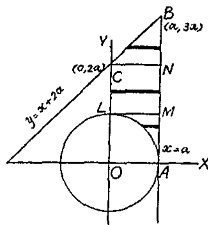
$$+\int_{a/2}^{a}\int_{y^2/a}^{a} f(x, y)\, dy\, dx.$$

for LMB

Ex. 4. *Change the order of integration in*

$$\int_0^a \int_{\sqrt{(a^2-x^2)}}^{x+2a} f(x, y)\, dx\, dy. \quad \text{(Agra 53, 58)}$$

The limits of integration give the region of integration to be bounded by $y=\sqrt{(a^2-x^2)}$, *i.e.* $x^2+y^2=a^2$, a circle with centre (0, 0) and radius a and the line $y=x+2a$ and $x=0$, $x=a$. Also $x=0$ meets the line in $(0, 2a)$ and $x=a$ meets



it at $(a, 3a)$. Now let us draw the figure and it is clear that the region of integration is $ABCLA$. Through L and C draw lines parallel to axis of x Now we shall divide the region into three parts in each of which the extremities of strips drawn parallel to axis of x lie on different sets of curves, namely region ALM, $LMNC$ and CNB.

Region. LAM. The extremities of strips parallel to axis of x lie on circle $x^2+y^2=a^2$ and the line $x=a$, so that the limits of x are $\sqrt{(a^2-y^2)}$ to a and the limits of y are $y=0$ for A and $y=a$ for L.

Hence after change of order the integration for the region LAM is $\int_0^a \int_{\sqrt{(a^2-y^2)}}^a f(x, y)\, dy\, dx.$ ...(1)

Region LMNC. The extremities of strips parallel to axis of x lie on $x=0$ and $x=a$ and hence the limits of x are 0 to a and the limits of y are clearly $y=a$ for L and $y=2a$ for C.

Hence after change of order the integration for the region $LMNC$ is $\int_0^{2a} \int_0^a f(x, y)\, dy\, dx.$ ...(2)

Region CNB. The extremities of strips parallel to axis of x lie on the line $y=x+2a$ and $x=a$, so that the limits of x are $y-2a$ to a. Also the limits of y are $y=2a$ for C and $y=3a$ for B.

Hence after change of order the integration for the region CNB is $\int_{2a}^{3a}\int_{x-2a}^{a} f(x, y)\, dy\, dx.$ ...(3)

Hence the given double integral = sum of integrals (1), (2) and (3)

Ex. 5. *Change the order of integration in*

$$\int_0^a \int_{\frac{1}{2}\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} V\, dx\, dy. \qquad \text{(Agra 51)}$$

The limits of integration give the region of integration to be bounded by

$y=\frac{1}{2}\sqrt{(a^2-x^2)}$ or $x^2+4y^2=a^2$

or $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2/4}=1,$

which is an ellipse of semi-axis a and $a/2$ and $y=\sqrt{(a^2-x^2)}$, *i.e.* $x^2+y^2=a^2$ and $x=0$, $x=a$. Now let us draw the figure and it is clear that the region of integration is $ACBA$.

Through B draw a line parallel to axis of x

Now we shall divide the region into two parts in each of which the extremities of strips parallel to axis of x lie on different sets of curves—namely regions ABM and BMC.

Region ABM. The extremities of strips parallel to axis of x lie on ellipse $x^2+4y^2=a^2$ and the circle $x^2+y^2=a^2$ and hence the limits of x are $\sqrt{(a^2-4y^2)}$ to $\sqrt{(a^2-y^2)}$ and the limits of y are $y=0$ for $y=a/2$ for B.

Hence after change of order the integration for region ABM is $\int_0^{a/2}\int_{\sqrt{(a^2-4y^2)}}^{\sqrt{(a^2-y^2)}} V\, dy\, dx.$ ...(1)

Hence after change of order, the integration for the region OAM is $\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\, dy\, dx$. ...(1)

Region AMB. The extremities of strips parallel to axis of x in the region lie on $x=0$ and $x^2+y^2=a^2$ so that limits of x are 0 to $\sqrt{(a^2-y^2)}$. Also the limits of y are given by $y=a\sin\alpha$ for A and $y=a$ for B.

Hence after change of order the integration for the region AMB is $\int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx$. ...(2)

Hence the given double integral=sum of integrals (1) and (2).

Verification. Let us choose $f(x, y)=1$; then the given double integral represents the area of a sector of circle in which angle $BOA=\frac{\pi}{2}-\alpha$ and hence its area is $\frac{1}{2}a^2\left(\frac{\pi}{2}-\alpha\right)$. Now we shall prove that sum of the integrals (1) and (2) is also $\frac{1}{2}a^2\left(\frac{\pi}{2}-\alpha\right)$. Now putting $f(x, y)=1$ in (1) and (2), we get

$$\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} dy\, dx$$

$$=\int_0^{a\sin\alpha}\Big[x\Big]_0^{y\cot\alpha} dy=\int_0^{a\sin\alpha} y\cot\alpha\, dy$$

$$=\cot\alpha.\frac{a^2\sin^2\alpha}{2}=\frac{1}{2}a^2\sin\alpha\cos\alpha \qquad ...(3)$$

and $$\int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} dy\, dx$$

$$=\int_{a\sin\alpha}^{a}\Big[x\Big]_0^{\sqrt{(a^2-y^2)}} dy=\int_{a\sin\alpha}^{a}\sqrt{(a^2-y^2)}\, dy$$

$$=\left[\frac{y}{2}\sqrt{(a^2-y^2)}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_{a\sin\alpha}^{a}$$

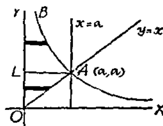
$$=[0-\tfrac{1}{2}a \sin \alpha\, a \cos \alpha]+\frac{a^2}{2}\left[\frac{\pi}{2}-\alpha\right]. \qquad \ldots(4)$$

Adding (3) and (4), we get $\frac{1}{2}a^2\left(\frac{\pi}{2}-\alpha\right)$, which is same as before. Hence proved.

Ex. 7. *Change the order of integration in*

$$\int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy$$

The limits of integration give the region of integration to be bounded by $y=x$ and $y=\frac{a^2}{x}$ *i.e.* $xy=a^2$ [*i.e.* a hyperbola whose vertex is (0, 0) and asymptotes the coordinate axes] and the lines $x=0$ and $x=a$. Hence the region of integration is $YOAB$ .., where A is (a, a). Through A draw a line parallel to axis of x dividing the region into two parts AOL and $YLAB$.... In each of these extremities of strips parallel to axis of x lie on different sets of curves



Region OAL. Extremities of strips lie on $x=0$ and $y=x$ so that the limits of x are 0 to y and those of y are clearly $y=0$ for O and $y=a$ for A. Hence after change of order, the integral for this region $=\int_0^a \int_0^x f(x, y)\, dy\, dx. \qquad \ldots(1)$

Region YALB... Extremities of strips lie on $x=0$ and $xy=a^2$ so that the limits of x are 0 to $\frac{a^2}{y}$ and those of y are clearly $y=a$ for A and $y=\infty$. Hence after change of order the integral for this region

$$=\int_0^\infty \int_0^{a^2/y} f(x, y)\, dy\, dx. \qquad \ldots(2)$$

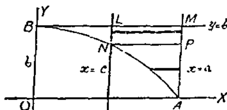
Hence the given integral = sum of integrals (1) and (2).

Ex. 8. *Change the order of integration in*

$$\int_c^a \int_{(b/a)\sqrt{(a^2-x^2)}}^{b} V\,dx\,dy \text{ where } c \text{ is less than } a.$$

The limits of integration give the region of integration to be bounded by $y=\frac{b}{a}\sqrt{(a^2-x^2)}$, *i.e.* ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $y=b$ and the lines $x=c$ and $x=a$ where c is less than a.

$y=b$ is a tangent to ellipse at B and $x=a$ is a tangent to ellipse at A. Draw the figure and we find that region of integration is given by $ANLMA$. Through N



draw a line parallel to axis of x thus dividing the region into two parts ANP and $NPML$ in each of which the extremities of strips drawn parallel to axis of x lie on different sets of curves. Putting $x=c$ in the ellipse, the point N is $\left\{c, \frac{b}{a}\sqrt{(a^2-c^2)}\right\}$. Also from the equation of ellipse, $x=\frac{a}{b}\sqrt{(b^2-y^2)}$.

Region ANP. In this region extremities of strips drawn parallel to axis of x lie on elli[illegible] $x=a$ so th[illegible] limits of x are from $\frac{a}{b}\sqrt{(b^2-y^2)}$ t[illegible] of [illegible] by $y=0$ for A and $y=\frac{b}{a}\sqrt{(a^2-}$[illegible]

Hence af[illegible] hange of [illegible] ANP is [illegible]

Region NPML. In this region the extremities lie on $x=c$ and $x=a$ so that the limits of x are from c to a and those of y are clearly $\frac{b}{a}\sqrt{(a^2-c^2)}$ for N and $y=b$ for M.

Hence after change of order the integration for the region $NPML$

$$=\int_{(b/a)\sqrt{(a^2-c^2)}}^{b} \int_{c}^{a} V\,dy\,dx. \qquad \ldots(2)$$

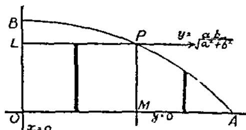
Hence the given integral=sum of integrals (1) and (2)

Ex. 9. *Change the order of integration in*

$$\int_{0}^{ab/\sqrt{(a^2+b^2)}} \int_{0}^{(a/b)\sqrt{(b^2-y^2)}} f(x, y)\,dy\,dx.$$

It should be clearly noted that given double integral is obtained by considering strips **parallel to axis of x** as indicated by $dy\,dx$ and in order to change the order, we will have to consider strips parallel to axis of y.

The region of integration is bounded by $x=0$ and $x=\frac{a}{b}\sqrt{(b^2-y^2)}$ *i.e.* ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the lines $y=0$ and $y=\frac{ab}{\sqrt{(a^2+b^2)}}$. Draw the figure and it is clear that the region of integration is given by $OAPLO$. Through P draw a line parallel to axis of y, thus dividing the region into two parts in each of which



the extremities of strips drawn parallel to axis of y lie on different sets of curves namely $OLPM$ and PMA.

Region OLPM. Here the extremities of strips parallel

to axis of y lie on $y=0$ and $y=\frac{ab}{\sqrt{(a^2+b^2)}}$ which give the limits of y and those of x are given by $x=0$ for O and $x=\frac{ab}{\sqrt{(a^2+b^2)}}$ for P $\left[\text{on solving the ellipse with } y=\frac{ab}{\sqrt{(a^2+b^2)}}\right.$ we get $\left. x=\frac{ab}{\sqrt{(a^2+b^2)}}\right]$.

Hence after change of order, the integration for this region is $\int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x, y)\, dx\, dy.$ ...(1)

Region MPA. In this region the extremities of strips parallel to axis of y lie on $y=0$ and ellipse $y=\frac{b}{a}\sqrt{(a^2-x^2)}$ which give the limits of y and those of x are given by $x=\frac{ab}{\sqrt{(a^2+b^2)}}$ for M and $x=a$ for A.

Hence after change of order the integration for this region is $\int_{ab/\sqrt{(a^2+b^2)}}^{a} \int_0^{(b/a)\sqrt{(a^2-x^2)}} f(x, y)\, dx\, dy$ ...(2)

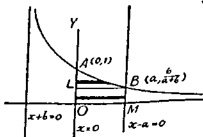
Hence the given integral = sum of integrals (1) and (2).

Ex. 10. *Change the order of integration in*

$$\int_0^a \int_0^{b/(b+x)} \phi(x, y)\, dx\, dy.$$

(Sagar 62 ; Agra 52)

Proceeding as usual, the region of integration is bounded by $y=0$ *i.e.* x-axis and $y=\frac{b}{b+x}$ or $xy+by-b=0$ [which is a rectangular hyperbola whose asymp-



totes are $y=0$ and $x=-b$ and vertex at $(-b, 0)$ and the lines $x=0$ and $x=a$.

Hence the region of integration is $ABMOA$. Through B draw a line parallel to x-axis, thus dividing the region into two parts $OMBL$ and LBA. Putting $x=0$ and $x=a$ in the rectangular hyperbola, the points A and B are given to be $(0, 1)$ and $\left(a, \frac{b}{a+b}\right)$ respectively.

Also from $y=\frac{b}{b+x}$, we get $x=\frac{b}{y}-b=b\,\frac{(1-y)}{y}$.

$\therefore$ Considering strips parallel to axis of x in each part,

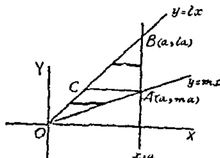
$$\text{Region } OMBL=\int_0^{b/(a+b)}\int_0^a \phi(x, y)\, dy\, dx. \qquad \ldots(1)$$

$$\text{Region } LBA=\int_{b/(a+b)}^1\int_0^{b(1-y)/y} \phi(x, y)\, dy\, dx. \qquad \ldots(2)$$

Hence the given integral after change of order=sum of integrals (1) and (2).

Ex. 11. *Change the order of integration in*

$$\int_0^a \int_{mx}^{lx} V\, dx\, dy.$$



Arguing as in various questions the whole region is bounded by $y=mx$, $y=lx$, and the lines $x=0$ and $x=a$. Through A draw a line parallel to axis of x dividing the region into two parts OAC and ACB. Considering

strips drawn parallel to axis of x in each region, we have

$$\text{Region } OAC = \int_0^{ma} \int_{y/l}^{y/m} V\,dy\,dx \qquad \ldots(1)$$

$$\text{Region } CAB = \int_{ma}^{la} \int_{y/l}^{a} V\,dy\,dx. \qquad \ldots(2)$$

Thus the given integral = sum of integrals (1) and (2).

Ex. 12. *Change the order of integration in*

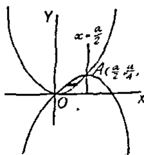
$$\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} V\,dx\,dy.$$

(Indore 66 ; Jiwaji 66 ; Vikram 62)

Proceeding as in various examples, *the limits of integration are* given by $y=\frac{x^2}{a}$, *i.e* parabola $x^2=ay$,

$$y=\frac{x-x^2}{a} \quad \text{or} \quad x^2-ax=-ay$$

or $$\left(x-\frac{a}{2}\right)^2=-a\left(y-\frac{a}{4}\right),$$



which represents a parabola whose *vertex is* $\left(\frac{a}{2}, \frac{a}{4}\right)$ *and concavity downwards* and it passes through origin. The two parabolas intersect at (0, 0), $\left(\frac{a}{2}, \frac{a}{4}\right)$.

Also the limits of x are given by $x=0$ and $x=\frac{a}{2}$.

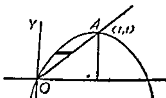
There are no separate parts as in other question.

Also $y=\frac{x^2}{a}$ gives $x=\sqrt{(ay)}$ and $x^2-ax+ay=0$ gives

$$x=\frac{a\pm\sqrt{(a^2-4ay)}}{2}.$$

Also from the equation of parabola,

$$x=\frac{2\pm\sqrt{(4-4y)}}{2}=1\pm\sqrt{(1-y)}.$$



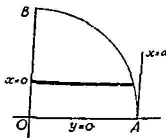
$y=x$ meets the parabola at (1, 1). Hence from figure, after change of order the integral is

$$\int_0^1\int_{1-\sqrt{(1-y)}}^{y} f(x, y)\, dy\, dx.$$

Ex. 15. *Change the order of integration in*

$$\int_0^a\int_0^{\sqrt{(a^2-x^2)}} f(x, y)\, dx\, dy.$$

Ans. $\displaystyle\int_0^a\int_0^{\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx.$



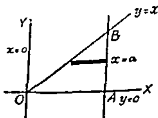
Ex. 16. *Change the order of integration in*

$$\int_0^a\int_0^x \frac{f'(y)\, dx\, dy}{\sqrt{[(a-x)\,(x-y)]}}$$

and hence find its value.

The region of integration is bounded by $y=0$, $y=x$ and $x=0$, $x=a$, *i.e.* region OAB. Considering strips parallel to axis of x after changing order, the given integral is

$$=\int_0^a\int_y^a \frac{f'(y)\, dy\, dx}{\sqrt{[(a-x)\,(x-y)]}} \qquad \ldots(1)$$



Value. Put $x=a\sin^2\theta+y\cos^2\theta$.

While integrating w. r. t. x, y is to be treated as constant.

$$\therefore\quad dx=2\,(a-y)\sin\theta\cos\theta\, d\theta.$$

Also $a-x=a\,(1-\sin^2\theta)-y\cos^2\theta=(a-y)\cos^2\theta$

and $\quad x-y=a\sin^2\theta-y\,(1-\cos^2\theta)=(a-y)\sin^2\theta.$

Also when $x=a$, then

$$(a-y)\cos^2\theta=0 \text{ or } \cos\theta=0\ ;\quad \therefore\quad \theta=\pi/2\ ;$$

and when $x=y$, then $(a-y)\sin^2\theta=0$ or $\sin\theta=0$; $\therefore\ \theta=0$,

$\because\ a-y$ is constant, it cannot be zero.

$$\int_y^a \frac{dx}{\sqrt{\{(a-x)(x-y)\}}}=\int_0^{\pi/2}\frac{2\,(a-y)\sin\theta\cos\theta\,d\theta}{(a-y)\sin\theta\cos\theta}$$

$$=2\int_0^{\pi/2} d\theta=2\Big[\theta\Big]_0^{\pi/2}=\pi.$$

Hence from (1), we have

$$I=\int_0^a\left\{f'(y)\int_y^a\frac{dx}{\sqrt{[(a-x)(x-y)]}}\right\}dy=\int_0^a f'(y).\pi\,dy$$

$$=\pi\Big[f(y)\Big]_0^a=\pi\,[f(a)-f(0)].$$

Ex. 17. *Change the order of integration in*

$$\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}}\frac{\phi'(y).(x^2+y^2)\,x\,dx\,dy}{\sqrt{[4a^2x^2-(x^2+y^2)^2]}}.$$

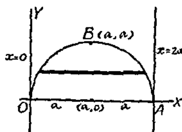
(Rajputana 62 ; Agra 67)

The region of integration is bounded by $y=0$ and $y=\sqrt{(2ax-x^2)}$ [*i.e.* circle

$$x^2+y^2-2ax=0$$

or $\quad (x-a)^2+y^2=a^2$

whose centre is $(a, 0)$ and radius a] and the lines $x=0$ and $x=2a$ *i.e.* the region OAB.



From the equation of circle, $x=\dfrac{2a\pm\sqrt{(4a^2-4y^2)}}{2}$

$$=a\pm\sqrt{(a^2-y^2)}.$$

Considering the strips parallel to x-axis whose extremities lie on the circle the limits of x are given to be from $a-\sqrt{(a^2-y^2)}$ to $a+\sqrt{(a^2-y^2)}$ and those of y are clearly $y=0$ for O and $y=a$ for B. Hence after change of order the given integral transforms to

$$\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} \frac{\phi'(y)\,(x^2+y^2)\,x\,dy\,dx}{\sqrt{[4a^2x^2-(x^2+y^2)^2]}}. \qquad \ldots(1)$$

Value. In evaluating integral (1), we have to first integrate

$$\int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} \frac{(x^2+y^2)\,x\,dx}{\sqrt{[4a^2x^2-(x^2+y^2)]}}.$$

Put $x^2=t$, $\therefore$ $2x\,dx=dt$, and let the limits of t be denoted by t_1 and t_2 for convenience sake.

or
$$I=\tfrac{1}{2}\int_{t_1}^{t_2} \frac{(t+y^2)\,dt}{\sqrt{[4a^2t-(t+y^2)^2]}}.$$

Now $4a^2t-(t+y^2)^2=-[(t^2+y^4+2ty^2-4a^2t+4a^4-4a^2y^2)$
$-4a^4+4a^2y^2]$

$$=4a^2(a^2-y^2)-(t+y^2-2a^2)^2.$$

Also $\quad t+y^2=(t+y^2-2a^2)+2a^2.$

$$\therefore \quad I=\tfrac{1}{2}\int_{t_1}^{t_2} \frac{(t+y^2-2a^2)\,dt}{\sqrt{[4a^2(a^2-y^2)-(t+y^2-2a^2)^2]}}$$
$$+\tfrac{1}{2}\,2a^2\int_{t_1}^{t_2} \frac{dt}{\sqrt{[4a^2(a^2-y^2)-(t+y^2-2a^2)^2]}}.$$

Remember that y is to be regarded as constant in the above integral and we know that

$$\tfrac{1}{2}\int \frac{z\,dz}{\sqrt{(a^2-z^2)}}=-\tfrac{1}{4}\int -\frac{2z\,dz}{\sqrt{(a^2-z^2)}}=-\tfrac{1}{4}\,2\sqrt{(a^2-z^2)}$$
$$=-\tfrac{1}{2}\sqrt{(a^2-z^2)}.$$

Also $\quad \displaystyle\int \frac{dz}{\sqrt{(a^2-z^2)}}=\sin^{-1}\frac{z}{a}.$

$$\therefore \quad I=-\tfrac{1}{2}\left[\sqrt{\{4a^2(a^2-y^2)-(t+y^2-2a^2)^2\}}\right]_{t_1}^{t_2}$$
$$+a^2\left[\sin^{-1}\frac{t+y^2-2a^2}{\sqrt{\{4a^2(a^2-y^2)\}}}\right]_{t_1}^{t_2}.$$

Now put $t=x^2$ and limits of x are as given

i.e. $\quad x=a-\sqrt{(a^2-y^2)}$

or $\quad x^2=a^2+a^2-y^2-2a\sqrt{(a^2-y^2)}$

or $\quad x^2+y^2-2a^2=-2a\sqrt{(a^2-y^2)} \qquad \ldots(1)$

When $x=a+\sqrt{(a^2-y^2)}$, then

$$x^2=a^2+a^2-y^2+2a\sqrt{(a^2-y^2)}$$

or

$$x^2+y^2-2a^2=2a\sqrt{(a^2-y^2)}. \quad \ldots(2)$$

For either limit $(x^2+y^2-2a^2)^2=4a^2\,(a^2-y^2) \quad \ldots(3)$

$$\therefore \quad I=-\tfrac{1}{2}\left[\sqrt{\left\{4a^2\,(a^2-y^2)-(x^2+y^2-2a^2)^2\right\}}\right]_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} + a^2\left[\sin^{-1}\frac{x^2+y^2-2a^2}{2a\sqrt{(a^2-y^2)}}\right]_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}}$$

$$=-\tfrac{1}{2}\left[0\right]+a^2\left[\sin^{-1}\frac{2a\sqrt{(a^2-y^2)}}{2a\sqrt{(a^2-y^2)}} - \sin^{-1}-\frac{2a\sqrt{(a^2-y^2)}}{2a\sqrt{(a^2-y^2)}}\right] \text{ by (1), (2) and (3).}$$

$$I=a^2\,\{\sin^{-1}1-\sin^{-1}(-1)\}=a^2\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=\pi a^2.$$

Hence the given integral $=\int_0^a \phi'(y)\,\pi a^2 . dy$

$$=\pi a^2\left[\phi(y)\right]_0^a=\pi a^2\,\{\phi(a)-\phi(0)\}.$$

Ex. 18. *Prove that*

$$\int_0^{\pi/2}\int_0^{\pi/2}\sin x\sin^{-1}(\sin x\sin y)\,dx\,dy=\frac{\pi}{2}\left(\frac{\pi}{2}-1\right).$$

(Agra 51)

Here we have to integrate w.r.t. y first treating x as constant.

$$\therefore \quad \text{Let } I_1=\int_0^{\pi/2}\sin^{-1}(\sin x\sin y)\,dy.$$

Put $\sin x\sin y=\sin z$, $\therefore$ $\sin x\cos y\,dy=\cos z\,dz$ (x constant).

$$\therefore \quad dy=\frac{\cos z\,dz}{\sin x\sqrt{(1-\sin^2 y)}}=\frac{\cos z\,dz}{\sqrt{(\sin^2 x-\sin^2 z)}}$$

$$=\frac{\cos z\,dz}{\sqrt{(\cos^2 z-\cos^2 x)}}.$$

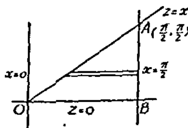
Also when $y=\pi/2$, then $\sin x=\sin z$, *i.e.* $z=x$

and when $y=0$, then $\sin z=0$; $\therefore$ $z=0$.

$$\therefore \quad I_1=\int_0^x \frac{z\cos z\,dz}{\sqrt{(\cos^2 z-\cos^2 x)}}.$$

$$\therefore \quad I=\int_0^{\pi/2}\int_0^x \frac{z\cos z\sin x\,dx\,dz}{\sqrt{(\cos^2 z-\cos^2 x)}}.$$

The region of integration is bounded by $z=0$ (*i.e.* $y=0$), $z=x$ and $x=0$, $x=\pi/2$. In order to change the order, we will have to consider strips parallel to axis of x, *i.e.* parallel to $y=0$, *i.e.* $z=0$. The extremities lie on $z=x$ and $x=\pi/2$ and hence the limits of x are z to $\pi/2$ and those of z are $z=0$ for O and $z=\pi/2$ for A.



Hence after changing of order, the integral takes the form

$$\int_0^{\pi/2}\int_z^{\pi/2}\frac{z\cos z\sin x}{\sqrt{(\cos^2 z-\cos^2 x)}}\,dz\,dx$$

$$=\int_0^{\pi/2} z\cos z\left[-\sin^{-1}\frac{\cos x}{\cos z}\right]_z^{\pi/2}dz$$

$$=\int_0^{\pi/2} z\cos z\,(-\sin^{-1}0+\sin^{-1}1)\,dz$$

$$=\frac{\pi}{2}\Big[z(\sin z)-(1)(-\cos z)\Big]_0^{\pi/2}$$

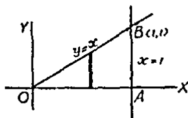
$$=\frac{\pi}{2}\left[\frac{\pi}{2}-1\right].$$

Proved.

Ex. 19. *When the region of integration is the triangle $y=0$, $y=x$ and $x=1$, show that*

$$I=\iint\sqrt{(4x^2-y^2)}\,dx\,dy=\tfrac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right).$$

Here we need not change the order. Since we are to integrate w.r.t. y treating x constant, we consider strips parallel to y-axis whose extremities lie on $y=0$ and $y=x$, so that the limits of y are given to be 0 to x and those of x are clearly $x=0$ for O and $x=1$ for A.



$$\therefore \quad I=\int_0^1 \int_0^x \sqrt{(4x^2-y^2)}\, dx\, dy$$

$$=\int_0^1 \left[\frac{y}{2}\sqrt{(4x^2-y^2)}+\frac{4x^2}{2}\sin^{-1}\frac{y}{2x}\right]_0^x dx$$

$$=\int_0^1 \left[\frac{x}{2}.x\sqrt{3}+2x^2.\frac{\pi}{6}\right) dx, \qquad \because \sin^{-1}\frac{1}{2}=\frac{\pi}{6}$$

$$=\int_0^1 \left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right) x^2\, dx=\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\right)\left[\frac{x^3}{3}\right]_0^1$$

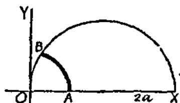
$$=\frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{3}\right). \qquad \text{Proved.}$$

Polar-Co-ordinates.

Ex. 20. *Change the order of integration in*

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, r\, d\theta\, dr.$$

The region of integration is bounded by $r=0$ and $r=2a\cos\theta$ (a circle on initial line as diameter of $2a$) and the lines $\theta=0$, *i.e.* initial line and $\theta=\pi/2$, *i.e.* OY. In the given form of integral, we have to first integrate w.r.t. r, *i.e.* we considered strips in which r was varying and θ constant just as in § 2 P. 161.



Now after change of order we will be integrating first w.r.t. θ, so that we have to consider strips of the type in which θ varies and r remains constant. Consider an elementary strip (circular type at a distance r from O) which has its extremities on $\theta=0$ and $r=2a\cos\theta$ (θ varying, r constant) and hence the limits of θ are 0 to $\cos^{-1} r/2a$. Also the limits of r are clearly $r=0$ for O and $r=2a$ for X.

Hence after change of order the given integral takes the form

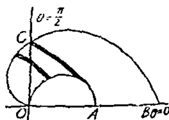
$$\int_0^{2a}\int_0^{\cos^{-1} r/2a} f(r, \theta)\, r\, dr\, d\theta.$$

Ex. 21. *Change the order of integration in*

$$\int_0^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r, \theta)\, r\, d\theta\, dr$$

$$+\int_{\pi/2}^{\pi}\int_0^{a(1+\cos\theta)} f(r, \theta)\, r\, d\theta\, dr.$$

$r=a(1+\cos\theta)$ is the well known equation of cardioid and $r=a\cos\theta$ is a circle on OA as diameter. The region of integration for the first integral is bounded by $r=a\cos\theta$, $r=a(1+\cos\theta)$



and $\theta=0$ and $\theta=\pi/2$, *i.e.* it corresponds to $ABCOAB$. The region of integration for the second integral is bounded by $r=0$ and $r=a(1+\cos\theta)$ and $\theta=\pi/2$ (for C), and $\theta=\pi$ (for O) *i.e.* it corresponds to $OCLOC$. Hence the total region given by both corresponds to $ABCLOAB$.

Now we shall consider strips in which r is constant and θ varying but before that we should divide the whole region into separate parts, so that in each part the extremities of

strips lie on different sets of curves. With O as centre and radius a *i.e.* OA draw a circular arc thus dividing the whole into two parts $ABCA$ and the part $ACLOA$ in each of which region we consider strips (r constant) whose extremities lie on different sets of curves.

Region ABCA. In this region the extremities of strips lie on $\theta=0$ and $r=a\,(1+\cos\theta)$, so that the limits of θ are from 0 to $\cos^{-1}\dfrac{r-a}{a}$ and the limits of r are from $r=a$ for A and $r=2a$ for B. Thus after change of order the integral for this part $=\displaystyle\int_0^{2a}\int_0^{\cos^{-1}(r-a)/a} f(r,\theta)\,r\,dr\,d\theta.$ ...(1)

Region ACLOA. In this region the extremities of strips lie on $r=a\cos\theta$ and $r=a\,(1+\cos\theta)$, so that the limits of θ are from $\cos^{-1}\dfrac{r}{a}$ to $\cos^{-1}\dfrac{r-a}{a}$. Also r varies from $r=0$ (at O) to $r=a$ for A. Thus after change of order the integral for this part

$$=\int_0^{a}\int_{\cos^{-1}(r/a)}^{\cos^{-1}(r-a)/a} f(r,\theta)\,r\,dr\,d\theta. \quad ...(2)$$

Hence after change of order the given integral
= sum of integrals (1) and (2).

§ 3. Transformation of Multiple Integrals.

Suppose we have to transform the given multiple integral

$$\iiint f(x, y, z)\,dx\,dy\,dz$$

to another system of variables u, v, w, where x, y, z are given as functions of u, v and w. This process of changing is called transformation of multiple integral.

Working Rule. (1) Change $f(x, y, z)$ by actual substitution of the values of x, y and z in terms of u, v, w and let the transformed function be $\phi(u, v, w)$.

(2) Determine the new limits of u, v, w. The help of geometrical conditions may be taken in determining the new limits.

(3) **To change dx dy dz (Important).**

The given multiple integral is

$$\int dx \int dy \int f(x, y, z)\, dz$$

and we know that while integrating w.r.t. z, x, y are to be treated as constant. We shall therefore express z in terms of x, y and w by the help of given relation *i.e.* **one old variable z in terms of one new variable w and two old variables x and y (Note).** Replace dz by $\frac{dz}{dw}\, dw$.

Again we shall express y in terms of v, w and x and replace dy by $\frac{dy}{dv}\, dv$, *i.e.* **2nd old variable y in terms of two new variables v, w and one old variable (but not z) which has already been eliminated.**

Lastly we shall express x in terms of u, v, w and replace dx by $\frac{dx}{du}\, du$, *i.e.* **3rd old variabl**[illegible] **of** [illegible]

Important. In general $\frac{dx}{du} \cdot \frac{dy}{dv} \cdot \frac{dz}{dw}$ is the Jacobian of x, y, z regarded as function of u, v, w,

$$i.e., \quad \frac{dx}{du} \cdot \frac{dy}{dv} \cdot \frac{dz}{dw} = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Hence we can directly write

$$dx\, dy\, dz = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du\, dv\, dw.$$

We shall illustrate the above by taking suitable examples.

Ex. 22. *Transform the integral* $\iint V\, dx\, dy$ *by the substitution* $x = r\cos\theta,\ y = r\sin\theta.$

From the given relation $x^2 + y^2 = r^2$.

$\therefore \quad y = \sqrt{(r^2 - x^2)}$, *i.e.*, one new variable r and x,

$x = r\cos\theta$, *i.e.*, two new variables r, θ.

$$dy = \frac{dy}{dr} \cdot dr = \frac{1}{2\sqrt{(r^2 - x^2)}} \cdot 2r\, dr = \frac{r}{y}\, dr = \frac{1}{\sin\theta}\, dr,$$

$$dx = \frac{dx}{d\theta}\, d\theta = -r\sin\theta\, d\theta.$$

$$\therefore \quad dx\, dy = \frac{dx}{d\theta} \cdot \frac{dy}{dr} \cdot d\theta\, dr = -r\sin\theta \cdot \frac{1}{\sin\theta}\, d\theta\, dr$$

$$= -r\, d\theta\, dr.$$

Also suppose V is transformed to V'.

Hence the given integral $= -\iint V'r\,d\theta\,dr$.

2nd Method. We could also write

$$x=\sqrt{(r^2-y^2)}\;;\quad \therefore\ \frac{dx}{dr}=\frac{r}{\sqrt{(r^2-y^2)}}=\frac{r}{x}=\frac{r}{r\cos\theta}=\frac{1}{\cos\theta}\;;$$

$$y=r\sin\theta,\qquad \therefore\ \frac{dy}{d\theta}=r\cos\theta.$$

$$\therefore\quad dx.dy=\frac{dx}{dr}.\frac{dy}{d\theta}.dr\,d\theta=\frac{1}{\cos\theta}\,r\cos\theta\,dr\,d\theta=r\,dr\,d\theta.$$

$$\therefore\quad \text{given integral}=\iint V'r\,dr\,d\theta.$$

3rd Method. Again

$$\frac{dx}{dr}.\frac{dy}{d\theta}=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial\theta}\\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial\theta}\end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r.$$

$$\therefore\quad dx\,dy=r\,dr\,d\theta.$$

Hence given integral $=\iint V'.r\,dr\,d\theta$

Ex. 23. *Transform the integral $\iiint V\,dx\,dy\,dz$ into new variables u, v, w, where $x+y+z=u$, $y+z=uv$, $z=uvw$.*

$x=u-(y+z)$ containing one new and two old.

$y=uv-z$ containing two new and one old (not x).

$z=uvw$ containing all the three new variables

$$\frac{dx}{du}=1,\quad \frac{dy}{dv}=u,\quad \frac{dz}{dw}=uv$$

$$\therefore \quad dx\, dy\, dz = \frac{dx}{du} \cdot \frac{dy}{dv} \cdot \frac{dz}{dw}\, du\, dv\, dw$$

$$= 1 \; u . uv \; du\, dv\, dw = u^2 v\; du\, dv\, dw$$

Let V be transformed to V' by putting x, y, z in terms of u, v, w,

i.e., $z = uvw$, $y = uv - z = uv - uvw$ and $x = u - (y+z) = u - uv$.

$\therefore$ the given integral $= \iiint V' . u^2 v . du\, dv\, dw$

Alternative. $dx\, dy\, dz = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}\, du\, dv\, dw$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} du\, dv\, dw$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v\,(1-w) & u\,(1-w) & -uv \\ vw & wu & uv \end{vmatrix} du\, dv\, dw.$$

Adding R_3 to R_2, we get

$$dx\, dy\, dz = \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & wu & uv \end{vmatrix} dx\, dy\, dz$$

$$= uv\, \{u\,(1-v) + uv\}\, du\, dv\, dw = u^2 v\; du\, dv\, dw.$$

Ex. 24. *Transform the integral* $\iiint V . dx\, dy\, dz$ *by the polar transformation*

$$x = r \sin\theta \sin\phi,\; y = r \sin\theta \cos\phi,\; z = r \cos\theta.$$

1st Method. $x^2+y^2=r^2 \sin^2 \theta$ and $x^2+y^2+z^2=r^2$.

$\therefore \quad x=\sqrt{(r^2-y^2-z^2)}$ containing one new variable and two old,

$z=r \cos \theta$ containing two new and no x,

$y=r \sin \theta \cos \phi$ containing three new variables and no x and z.

$$\frac{dx}{dr}=\frac{1.2r}{2\sqrt{(r^2-y^2-z^2)}}=\frac{r}{x}=\frac{r}{r \sin \theta \sin \phi}=\frac{1}{\sin \theta \sin \phi}.$$

$$\frac{dz}{d\theta}=-r \sin \theta \text{ and } \frac{dy}{d\phi}=-r \sin \theta \sin \phi.$$

$$\therefore \quad dx\,dy\,dz=\frac{dx}{dr}\cdot\frac{dz}{d\theta}\cdot\frac{dy}{d\phi}\,dr\,d\theta\,d\phi$$

$$=\frac{1}{\sin \theta \sin \phi}(-r \sin \theta)(-r \sin \theta \sin \phi)\,dr\,d\theta\,d\phi$$

$$=r^2 \sin \theta\,dr\,d\theta\,d\phi.$$

Let V' be the corresponding value of V; when changed to polars, then the given integral$=\iiint V' r^2 \sin \theta\,dr\,d\theta\,d\phi$.

Alternative.

$$dx\,dy\,dz=\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\,dr\,d\theta\,d\phi$$

$$=\begin{vmatrix} \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} dr\,d\theta\,d\phi$$

$$=[r \sin \theta \cos \phi\,\{-r \cos \phi\,(\sin^2 \theta+\cos^2 \theta)\} +r \sin \theta \sin \phi\,\{-r \sin \phi\,(\sin^2 \theta+\cos^2 \theta)\}]\,dr\,d\theta\,d\phi$$

$$=[-r^2 \sin \theta\,(\sin^2 \phi+\cos^2 \phi)]\,dr\,d\theta\,d\phi=-r^2 \sin \theta\,dr\,d\theta\,d\phi$$

etc.

Ex. 25. *Transform the multiple integral*

$$\iiint V\,dx_1\,dx_2\,dx_3\,dx_4,$$

where $\quad x_1=r\sin\theta\cos\phi,\quad x_2=r\cos\theta\cos\psi,$

$$x_3=r\sin\theta\sin\phi,\quad x_4=r\cos\theta\sin\psi,$$

$$x_1^2+x_3^2=r^2\sin^2\theta,\ x_2^2+x_4^2=r^2\cos^2\theta.$$

$$\therefore\quad x_1^2+x_2^2+x_3^2+x_4^2=r^2.$$

$x_1=\sqrt{(r^2-x_2^2-x_3^2-x_4^2)}$ containing one new and three old $x_2, x_3, x_4,$

$x_2=\sqrt{(r^2\cos^2\theta-x_4^2)}$ containing two new and no x_1,

$x_3=r\sin\theta\sin\phi$ containing three new and no x_1, x_2,

$x_4=r\cos\theta\sin\psi$ all new and no x_1, x_2, x_3.

$$\frac{dx_1}{dr}=\frac{r}{x_1}=\frac{1}{\sin\theta\cos\phi},$$

$$\frac{dx_2}{d\theta}=-\frac{2r^2\cos\theta\sin\theta}{x_2}=-\frac{2r^2\cos\theta\sin\theta}{r\cos\theta\cos\psi}=-\frac{2r\sin\theta}{\cos\psi},$$

$$\frac{dx_3}{d\phi}=r\sin\theta\cos\phi \text{ and } \frac{dx_4}{d\psi}=r\cos\theta\cos\psi.$$

$$\therefore\quad dx_1\,dx_2\,dx_3\,dx_4=\frac{2}{\sin\theta\cos\phi}\left(\frac{-2r\sin\theta}{\cos\psi}\right)(r\sin\theta\cos\phi)\times(r\cos\theta\cos\psi)\,dr\,d\theta\,d\phi\,d\psi$$

$$=-r^3\sin\theta\cos\theta.dr\,d\theta\,d\phi\,d\psi.$$

If V' be the corresponding value of V after converting to polar, then the given integral transforms to

$$-\iiiint V'r^3\sin\theta\cos\theta\,dr\,d\theta\,d\phi\,d\psi.$$

Note. You may evaluate yourself by the help of Jacobian.

Ex. 26. *Transform the double integral* $\iint x^{m-1}y^{n-1}\,dx\,dy$ *by the formula* $x+y=u,\ y=uv.$

$$x=u-y \text{ and } y=uv\ ;\quad \therefore\quad \frac{dx}{du}=1,\ \frac{dy}{dv}=u.$$

$\therefore \quad dx\,dy = \dfrac{dx}{du} \cdot \dfrac{dy}{dv}\,du\,dv = u\,du\,dv.$

$$y = uv \text{ and } x = u - uv = u\,(1-v).$$

$\therefore \quad x^{m-1}y^{n-1} = u^{m-1}\,(1-v)^{m-1}\,u^{n-1}v^{n-1} = u^{m+n-2}v^{n-1}\,(1-v)^{m-1}.$

$\therefore$ The given integral $= \iint u^{m+n-2}v^{n-1}\,(1-v)^{m-1}\,u\,du\,dv$

$$= \iint u^{m+n-1}v^{n-1}\,(1-v)^{m-1}\,du\,dv.$$

Ex. 27. *Transform the integral* $\iint V\,dx\,dy\,dz.$

(Rajputana 62)

If $\quad x = \dfrac{u_2u_3}{u_1},\ \ y = \dfrac{u_3u_1}{u_2},\ \ z = \dfrac{u_1u_2}{u_3},$

$$dx\,dy\,dz = \frac{\partial\,(x, y, z)}{\partial\,(u_1, u_2, u_3)}\,du_1,\ du_2,\ du_3.$$

Now $\dfrac{\partial\,(x, y, z)}{\partial\,(u_1, u_2, u_3)} = \begin{vmatrix} -\dfrac{u_2u_3}{u_1^2} & \dfrac{u_3}{u_1} & \dfrac{u_2}{u_1} \\ \dfrac{u_3}{u_2} & -\dfrac{u_3u_1}{u_2^2} & \dfrac{u_1}{u_2} \\ \dfrac{u_2}{u_3} & \dfrac{u_1}{u_3} & -\dfrac{u_1u_2}{u_3^2} \end{vmatrix}$

$$= \frac{1}{u_1u_2u_3}\begin{vmatrix} -\dfrac{u_2u_3}{u_1} & u_3 & u_2 \\ u_3 & -\dfrac{u_3u_1}{u_2} & u_1 \\ u_2 & u_1 & -\dfrac{u_1u_2}{u_3} \end{vmatrix}$$

$$= \frac{1}{u_1u_2u_3}\left[-\frac{u_2u_3}{u_1}\left(\frac{u_3u_1}{u_2}\cdot\frac{u_1u_2}{u_3} - u_1^2\right) - u_3\left(-\frac{u_1u_2u_3}{u_3} - u_1u_2\right) + u_2\left(u_1u_3 + \frac{u_3u_1u_2}{u_2}\right)\right]$$

$$=\frac{1}{u_1u_2u_3}.[0+2u_1u_2u_3+2u_1u_2u_3]=4.$$

$$\therefore \quad dx\ dy\ dz=4\ du_1\ du_2\ du_3.$$

Let V become V' after substitution of new variables.

Hence the given integral

$$=\int V'.4\ du\ dv\ dw=4\int V_1\ du\ dv\ dw.$$

Ex. 28. *Transform the integral* $\iiint V\ dx\ dy\ dz$, *when*

$x=r\sin\theta\sqrt{(1-m^2\sin^2\phi)},\ y=r\sin\phi\sqrt{(1-n^2\sin^2\theta)},$

$z=r\cos\theta\cos\phi$ *and* $m^2+n^2=1$.

$$dx\ dy\ dz=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\,dr\ d\theta\ d\phi,$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}=\Delta \text{ say.}$$

From the given relations,

$$x^2+y^2=r^2[\sin^2\theta+\sin^2\phi-(m^2+n^2)\sin^2\theta\sin^2\phi].$$

$$\therefore\quad x^2+y^2+z^2=r^2[\cos^2\theta\cos^2\phi+\sin^2\theta+\sin^2\phi-\sin^2\theta\sin^2\phi]$$

$$\because\ m^2+n^2=1$$

$$=r^2[(1-\sin^2\theta)(1-\sin^2\phi)+\sin^2\theta+\sin^2\phi-\sin^2\theta\sin^2\phi]$$

$$=r^2[1-\sin^2\theta-\sin^2\phi+\sin^2\theta+\sin^2\phi]=r^2.$$

Differentiating $x^2+y^2+z^2=r^2$ partially w.r.t. r, θ and ϕ, we get

$$x\frac{\partial x}{\partial r}+y\frac{\partial y}{\partial r}+z\frac{\partial z}{\partial r}=r,$$

$$x\frac{\partial x}{\partial \theta}+y\frac{\partial y}{\partial \theta}+z\frac{\partial z}{\partial \theta}=0,$$

$$x\frac{\partial x}{\partial \phi}+y\frac{\partial y}{\partial \phi}+z\frac{\partial z}{\partial \phi}=0.$$

Now keeping in view the above relations, we multiply the first row of Δ by x and hence divide Δ by x and then add to this first row y times 2nd row and z times third row.

$$\therefore \quad \Delta=\frac{1}{x}\begin{vmatrix} \Sigma x\frac{\partial x}{\partial r} & \Sigma x\frac{\partial x}{\partial \theta} & \Sigma x\frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}=\frac{1}{x}\begin{vmatrix} r & 0 & 0 \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$=\frac{r}{x}\left(\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\frac{\partial z}{\partial \theta}\right)$$

$$=\frac{r}{x}\left[\frac{r\sin\phi\,(-2n^2\sin\theta\cos\theta)}{2\sqrt{(1-n^2\sin^2\theta)}}(-r\cos\theta\sin\phi)\right.$$

$$\left.-r\cos\phi\sqrt{(1-n^2\sin^2\theta)}\,(-r\sin\theta\cos\phi)\right]$$

$$=\frac{r}{r\sin\theta\sqrt{(1-m^2\sin^2\phi)}}$$

$$\times\left[\frac{r^2.\sin\theta}{\sqrt{(1-n^2\sin^2\theta)}}\{n^2\sin^2\phi\cos^2\theta+\cos^2\phi\,(1-n^2\sin^2\theta)\}\right]$$

Put $1=m^2+n^2$.

$\therefore \; 1-n^2\sin^2\theta=m^2+n^2(1-\sin^2\theta)=m^2+n^2\cos^2\theta.$

$$=\frac{r^2}{\sqrt{(1-m^2\sin^2\phi)}\sqrt{(1-n^2\sin^2\theta)}}$$

$$\times[n^2\sin^2\phi\cos^2\theta+m^2\cos^2\phi+n^2\cos^2\phi\cos^2\theta]$$

$$=\frac{r^2}{\sqrt{(1-m^2\sin^2\phi)}\sqrt{(1-n^2\sin^2\theta)}}$$

$$\times[n^2\cos^2\theta+m^2\cos^2\phi].$$

If V' be the value of V after transformation in terms of new variables, then the given integral$=\iint V'.\Delta.dr\,d\theta\,d\phi$, where Δ has the value written above.

Ex. 29. *Prove that if*

$$x=u\sin\alpha+v\cos\alpha,\ y=u\cos\alpha-v\sin\alpha,$$

$$\iint\frac{f(x,y)}{\sqrt{(1-x^2-y^2)}}\,dx\,dy=-\left[\int\frac{\phi(u,v)\,du\,dv}{\sqrt{(1-u^2-v^2)}}\right.$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\sin\alpha & \cos\alpha\\ \cos\alpha & -\sin\alpha\end{vmatrix}=-1.$$

$$\therefore\quad dx\,dy=\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv=-du\,dv.$$

Also $x^2+y^2=u^2(\sin^2\alpha+\cos^2\alpha)+v^2(\cos^2\alpha+\sin^2\alpha)$

$$=u^2+v^2.$$

If $f(x,y)$ becomes $\phi(u,v)$, then

$$I=-\int\frac{\phi(u,v).du\,dv}{\sqrt{(1-u^2-v^2)}}.$$

Ex. 30. *Transform to polar co-ordinates and integrate* $\iint\sqrt{\left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)}\,dx\,dy$, *the integral being extended over all positive values of x and y subject to* $x^2+y^2\leqslant 1$. **(Agra 45)**

For transformation, $x=r\cos\theta,\ y=r\sin\theta$.

Now $\quad dx\,dy=r\,dr\,d\theta$. [See Ex. 22 P. 187 for proof]

Now in the +ive octant of $x^2+y^2\leqslant 1$, θ varies from 0 to 90 and r varies from 0 to 1.

Also $\quad x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)=r^2$,

$$\therefore\quad I=\int_0^1\int_0^{\pi/2}\sqrt{\left(\frac{1-r^2}{1+r^2}\right)}.r\,dr\,d\theta$$

$$=\int_0^1 r\sqrt{\left(\frac{1-r^2}{1+r^2}\right)}\,dr.\Big[\theta\Big]_0^{\pi/2}$$

$$=\frac{\pi}{2}\int_0^1 \frac{r\,(1-r^2)}{\sqrt{(1-r^4)}}\,dr.$$ Put $r^2=\sin\theta$ and adjust the limits.

$$=\frac{\pi}{2}\,\frac{1}{2}\int_0^{\pi/2}\frac{(1-\sin\theta)}{\cos\theta}\cos\theta\,d\theta=\frac{\pi}{4}\Big[\theta+\cos\theta\Big]_0^{\pi/2}$$

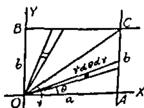
$$=\frac{\pi}{4}\left(\frac{\pi}{2}-1\right)$$

Ex. 31. *Find the value of*

$$\int_0^a\int_0^b \frac{dx\,dy}{(c^2+x^2+y^2)^{3/2}}$$ *by transforming it to polars.*

Putting $x=r\cos\theta$, $y=r\sin\theta$, we have $dx\,dy=r\,d\theta\,dr$.

The limits of integration suggest that the region of integration is bounded by $y=0$, $y=b$ and $x=0$, $x=a$, *i.e.* a rectangle.



Hence the given integral after transformation is

$$\iint \frac{r\,d\theta\,dr}{(c^2+r^2)^{3/2}}$$

taken over the rectangle $OACB$ in which $OA=a$ and $AC=b$. Join OC thus dividing the region of integration into triangles OAC and OCB, where $\tan\gamma=\frac{b}{a}$.

$$\therefore\quad \sin\gamma=\frac{b}{\sqrt{(a^2+b^2)}}\quad\text{and}\quad\cos\gamma=\frac{a}{\sqrt{(a^2+b^2)}}.$$

Region OAC. The limits of r are clearly from 0 to $a\sec\theta$ and those of θ are $\theta=0$ to $\theta=\gamma$.

Region OCB. The limits of r in this region are from 0 to $b\operatorname{cosec}\theta$ and those of θ are from γ to $\frac{\pi}{2}$. Hence the above integral

$$=\int_0^{\gamma}\int_0^{a\sec\theta}\frac{r\,d\theta\,dr}{(r^2+c^2)^{3/2}}+\int_{\gamma}^{\pi/2}\int_0^{b\operatorname{cosec}\theta}\frac{r\,d\theta\,dr}{(r^2+c^2)^{3/2}}=I_1+I_2.$$

Now $\int \frac{r\,dr}{(r^2+c^2)^{3/2}}=\frac{1}{2}\int \frac{2r\,dr}{(r^2+c^2)^{3/2}}$

$$=\frac{1}{2}\cdot\frac{1}{-\frac{1}{2}}\,\frac{1}{(r^2+c^2)^{1/2}}=-\frac{1}{\sqrt{(r^2+c^2)}}.$$

$$\therefore\quad I_1=\int_0^{\gamma}-\left[\frac{1}{\sqrt{(r^2+c^2)}}\right]_0^{a\sec\theta}d\theta$$

$$=\int_0^{\gamma}\left(\frac{1}{c}-\frac{1}{\sqrt{(a^2\sec^2\theta+c^2)}}\right)d\theta$$

$$=\frac{\gamma}{c}-\int_0^{\gamma}\frac{\cos\theta\,d\theta}{\sqrt{(a^2+c^2-c^2\sin^2\theta)}}\,d\theta$$

$$=\frac{\gamma}{c}-\frac{1}{c}\left[\sin^{-1}\frac{c\sin\theta}{\sqrt{(a^2+c^2)}}\right]_0^{\gamma}$$

$$=\frac{\gamma}{c}-\frac{1}{c}\sin^{-1}\frac{c\sin\gamma}{\sqrt{(a^2+c^2)}}$$

$$=\frac{\gamma}{c}-\frac{1}{c}\sin^{-1}\frac{bc}{\sqrt{(a^2+c^2)}\sqrt{(a^2+b^2)}}.$$

Now $\qquad \sin^{-1}\frac{x}{a}=\tan^{-1}\frac{x}{\sqrt{(a^2-x^2)}}.$

$$\therefore\quad \sin^{-1}\frac{bc}{\sqrt{(a^2+c^2)}\sqrt{(a^2+b^2)}}=\tan^{-1}\frac{bc}{a\sqrt{(a^2+b^2+c^2)}}.$$

$$I_1=\frac{\gamma}{c}-\frac{1}{c}\tan^{-1}\frac{bc}{a\sqrt{(a^2+b^2+c^2)}}. \qquad \ldots(1)$$

$$I_2=\int_{\gamma}^{\pi/2}-\left[\frac{1}{\sqrt{(r^2+c^2)}}\right]_0^{b\,\mathrm{cosec}\,\theta}d\theta$$

$$=\int_{\gamma}^{\pi/2}\left\{\frac{1}{c}-\frac{1}{\sqrt{(b^2\,\mathrm{cosec}^2\,\theta+c^2)}}\right\}d\theta$$

$$=\frac{1}{c}\left(\frac{\pi}{2}-\gamma\right)-\int_{\gamma}^{\pi/2}\frac{\sin\theta\,d\theta}{\sqrt{(b^2+c^2-c^2\cos^2\theta)}}$$

$$=\frac{1}{c}\left(\frac{\pi}{2}-\gamma\right)+\frac{1}{c}\left[\sin^{-1}\frac{c\cos\theta}{\sqrt{(b^2+c^2)}}\right]_{\gamma}^{\pi/2}$$

$$=\frac{1}{c}\left(\frac{\pi}{2}-\gamma\right)+\frac{1}{c}\left\{0-\sin^{-1}\frac{c\cos\gamma}{\sqrt{(b^2+c^2)}}\right\}\quad \text{Put for } \cos\gamma$$

$$=\frac{1}{c}\left(\frac{\pi}{2}-\gamma\right)-\frac{1}{c}\sin^{-1}\frac{ac}{\sqrt{(b^2+c^2)}\sqrt{(b^2+a^2)}}$$

$$=\frac{1}{c}\left(\frac{\pi}{2}-\gamma\right)-\frac{1}{c}\tan^{-1}\frac{ac}{b\sqrt{(a^2+b^2+c^2)}}. \qquad \ldots(2)$$

Adding (1) and (2), we have

$$I_1+I_2=\frac{1}{c}\left[\gamma+\frac{\pi}{2}-\gamma-\tan^{-1}\frac{bc}{a\sqrt{(a^2+b^2+c^2)}}\right.$$
$$\left.-\tan^{-1}\frac{ac}{b\sqrt{(a^2+b^2+c^2)}}\right]$$

Now apply $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$

$$=\frac{1}{c}\left\{\frac{\pi}{2}-\tan^{-1}\frac{\frac{c}{\sqrt{(a^2+b^2+c^2)}}\left(\frac{a^2+b^2}{ab}\right)}{\left(1-\frac{c^2}{a^2+b^2+c^2}\right)}\right\}$$

$$=\frac{1}{c}\left[\frac{\pi}{2}-\tan^{-1}\frac{c\sqrt{(a+b^2+c^2)}}{ab}\right]=\frac{1}{c}\cot^{-1}\frac{c\sqrt{(a^2+b^2+c^2)}}{ab}$$

$$=\frac{1}{c}\tan^{-1}\frac{ab}{c\sqrt{(a^2+b^2+c^2)}}. \quad \because \quad \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}.$$

Ex. 32. *Show by polar transformation that*

$$c\int_0^{c\tan\alpha/\sqrt{2}}\int_0^{c\tan\alpha/\sqrt{2}}\frac{dx\,dy}{(x^2+y^2+c^2)^{3/2}}$$
$$=\tan^{-1}\frac{\sec\alpha-\cos\alpha}{2}=\tan^{-1}\frac{\sin^2\alpha}{2\cos\alpha}.$$

It is exactly the same question as before. For convenience sake put $a=b=c\tan\alpha/\sqrt{2}$, so that the rectangle of last question becomes square and $\tan\gamma=b/a=1$. *i.e.* $\gamma=\pi/4$. Hence after transformation the given integral

$$=c\int_0^{\pi/4}\int_0^{a\sec\theta}\frac{r\,d\theta\,dr}{(r^2+c^2)^{3/2}}+c\int_{\pi/4}^{\pi/2}\int_0^{a\,\mathrm{cosec}\,\theta}\frac{r\,d\theta\,dr}{(r^2+c^2)^{3/2}}$$
$$=c\,(I_1+I_2).$$

$$I_1=\int_0^{\pi/4}-\left[\frac{1}{\sqrt{(r^2+c^2)}}\right]_0^{a\sec\theta}d\theta=\int_0^{\pi/4}\left(\frac{1}{c}-\frac{1}{\sqrt{(a^2\sec^2\theta+c^2)}}\right)d\theta$$

$$=\frac{\pi}{4c}-\int_0^{\pi/4}\frac{\cos\theta\,d\theta}{\sqrt{(a^2+c^2-c^2\sin^2\theta)}}=\frac{\pi}{4c}-\frac{1}{c}\left[\sin^{-1}\frac{c\sin\theta}{\sqrt{(a^2+c^2)}}\right]_0^{\pi/4}$$

$$=\frac{\pi}{4c}-\frac{1}{c}\sin^{-1}\frac{c}{\sqrt{2}\sqrt{(a^2+c^2)}}=\frac{\pi}{4c}-\frac{1}{c}\tan^{-1}\frac{c}{\sqrt{[2(a^2+c^2)-c^2]}}$$

$$=\frac{\pi}{4c}-\frac{1}{c}\tan^{-1}\frac{c}{\sqrt{(2a^2+c^2)}},\text{ Put } a=\frac{c\tan\alpha}{\sqrt{2}},\ i.e.\ 2a^2=c^2\tan^2\alpha.$$

$$\therefore\quad I_1=\frac{\pi}{4c}-\frac{1}{c}\tan^{-1}\frac{c}{\sqrt{[c^2(1+\tan^2\alpha)]}}=\frac{\pi}{4c}-\frac{1}{c}\tan^{-1}\cos\alpha.$$

$$I_2=\int_{\pi/4}^{\pi/2}-\left[\frac{1}{\sqrt{(r^2+c^2)}}\right]_0^{a\operatorname{cosec}\theta}$$

$$=\int_{\pi/4}^{\pi/2}\left[\frac{1}{c}-\frac{1}{\sqrt{(a^2\operatorname{cosec}^2\theta+c^2)}}\right]d\theta$$

$$=\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\frac{1}{c}-\int_{\pi/4}^{\pi/2}\frac{\sin\theta}{\sqrt{(a^2+c^2-c^2\cos^2\theta)}}\,d\theta$$

$$=\frac{\pi}{4c}+\frac{1}{c}\left[\sin^{-1}\frac{c\cos\theta}{\sqrt{(a^2+c^2)}}\right]_{\pi/4}^{\pi/2}$$

$$=\frac{\pi}{4c}+\frac{1}{c}\left[0-\sin^{-1}\frac{c}{\sqrt{2}\sqrt{(a^2+c^2)}}\right]$$

$$=\frac{\pi}{4c}-\frac{1}{c}\tan^{-1}\cos\alpha\quad\text{as in } I_1.$$

$$\therefore\quad c\,(I_1+I_2)$$

$$=\left(\frac{\pi}{4}+\frac{\pi}{4}\right)-2\tan^{-1}\cos\alpha=\frac{\pi}{2}-\tan^{-1}\frac{2\cos\alpha}{1-\cos^2\alpha}$$

$$=\cot^{-1}\frac{2\cos\alpha}{1-\cos^2\alpha}=\tan^{-1}\frac{1-\cos^2\alpha}{2\cos\alpha}=\tan^{-1}\frac{\sec\alpha-\cos\alpha}{2}$$

or $=\tan^{-1}\dfrac{\sin^2\alpha}{2\cos\alpha}$.

Ex. 33. *Show by transformation to polar co-ordinates that* $\displaystyle\int_0^{a\tan\alpha}\int_0^{a\tan\beta}\frac{dx\,dy}{(x^2+y^2+a^2)^2}$

$$=\frac{1}{2a^2}\{\sin\alpha\tan^{-1}(\tan\beta\cos\alpha)+\sin\beta\tan^{-1}(\tan\alpha\cos\beta)\}.$$

(Agra 61)

Let us put $a \tan \alpha = A$ and $a \tan \beta = B$; so the region of integration is bounded by rectangle $x=0$, $y=0$, $x=A$, $y=B$. Let us suppose that $\tan \gamma = \frac{B}{A} = \frac{\tan \beta}{\tan \alpha}$ so that after converting into polar, it is $\iint \frac{r\, d\theta\, dr}{(r^2+a^2)^2}$, and it can be taken as sum of two integrals as in Q. 31.

$$=\int_0^{\gamma} \int_0^{A \sec \theta} \frac{r\, d\theta\, dr}{(r^2+a^2)^2} + \int_{\gamma}^{\pi/2} \int_0^{B \operatorname{cosec} \theta} \frac{r\, d\theta\, dr}{(r^2+a^2)^2} = I_1 + I_2.$$

Now $\int \frac{r\, dr}{(r^2+a^2)^2} = \frac{1}{2} \int \frac{2r\, dr}{(r^2+a^2)^2} = -\frac{1}{2(r^2+a^2)}$.

$$\therefore \quad I_1 = \int_0^{\gamma} -\frac{1}{2} \left[\frac{1}{r^2+a^2} \right]_0^{A \sec \theta} = -\frac{1}{2} \int_0^{\gamma} \left(\frac{1}{A^2 \sec^2 \theta + a^2} - \frac{1}{a^2} \right) d\theta$$

$$= \frac{1}{2a^2} \int_0^{\gamma} \frac{A^2 \sec^2 \theta}{A^2 \sec^2 \theta + a^2}\, d\theta = \frac{1}{2a^2} \int_0^{\gamma} \frac{A^2 \sec^2 \theta}{(A^2+a^2) + A^2 \tan^2 \theta}\, d\theta$$

$$= \frac{A}{2a^2} \cdot \frac{1}{\sqrt{(A^2+a^2)}} \left[\tan^{-1} \frac{A \tan \theta}{\sqrt{(A^2+a^2)}} \right]_0^{\gamma}.$$

Put $A = a \tan \alpha$. $\therefore \quad A^2 + a^2 = a^2 \sec^2 \alpha$,

$$\therefore \quad I_1 = \frac{a \tan \alpha}{2a^2 . a \sec \alpha} \tan^{-1} \frac{a \tan \alpha \tan \gamma}{a \sec \alpha}$$

$$= \frac{1}{2a^2} \sin \alpha \tan^{-1} (\tan \beta \cos \alpha) \text{ by (2)}. \quad \ldots(1)$$

$$\because \quad \tan \gamma = \frac{\tan \beta}{\tan \alpha} \text{ or } \tan \alpha \tan \gamma = \tan \beta \text{ or } \tan \beta \cot \gamma = \tan \alpha. \quad \ldots(2)$$

Similarly we can find the value of I_2.

$$I_2 = \int_{\gamma}^{\pi/2} -\frac{1}{2} \left[\frac{1}{r^2+a^2} \right]_0^{B \operatorname{cosec} \theta} = -\frac{1}{2} \int_{\gamma}^{\pi/2} \left[\frac{1}{B^2 \operatorname{cosec}^2 \theta + a^2} - \frac{1}{a^2} \right] d\theta$$

$$= \frac{1}{2a^2} \int_{\gamma}^{\pi/2} \frac{B^2 \operatorname{cosec}^2 \theta}{B^2 + a^2 + B^2 \cot^2 \theta}\, d\theta = -\frac{B}{2a^2} \int_{\gamma}^{\pi/2} \frac{-B \operatorname{cosec}^2 \theta\, d\theta}{B^2 + a^2 + B^2 \cot^2 \theta}$$

$$= -\frac{B}{2a^2} \cdot \frac{1}{\sqrt{(B^2+a^2)}} \left[\tan^{-1} \frac{B \cot \theta}{\sqrt{(B^2+a^2)}} \right]_{\gamma}^{\pi/2}.$$

Put $B = a \tan \beta$ $\therefore$ $\ldots = a^2 \sec^2 \beta$.

$$=-\frac{a \tan \beta}{2a^2 . a \sec \beta}\left[0-\tan^{-1} \frac{a \tan \beta \cot \gamma}{a \sec \beta}-0\right]. \quad \because \quad \cot 90=0$$

$$=\frac{1}{2a^2} \sin \beta \tan^{-1} (\tan \alpha \cos \beta) \text{ by (2).} \qquad \ldots(3)$$

Hence the given integral

$$=\frac{1}{2a^2} [\sin \alpha \tan^{-1} (\cos \alpha \tan \beta)+\sin \beta \tan^{-1} (\cos \beta \tan \alpha)]$$

Ex. 34. *Transform* $\int_0^\infty \int_0^\infty e^{-(x^2+2xy \cos \alpha+y^2)}\, dx\, dy$ *from rectangular to polar co-ordinates, and hence show that its integral is* $\frac{\alpha}{2 \sin \alpha}$.

$x=r \cos \theta$, $y=r \sin \theta$; $\therefore$ $x^2+y^2=r^2$ and $dx\, dy=r\, d\theta\, dr$.

Evidently the limits of integration extend from $x=0$ to $x=\infty$ and $y=0$ to $y=\infty$.

The limits of r therefore extend from 0 to ∞ and those of θ from 0 to $\frac{\pi}{2}$ in the +ive quadrant.

$$\therefore \quad I=\int_0^{\pi/2} \int_0^\infty e^{-(r^2+2r^2 \sin \theta \cos \theta \cos \alpha)}\, r\, d\theta\, dr$$

$$=-\frac{1}{2}\int_0^{\pi/2} \int_0^\infty e^{-r^2 (1+2 \sin \theta \cos \theta \cos \alpha)} (-2r\, d\theta\, dr)$$

$$=-\frac{1}{2}\int_0^{\pi/2} \left[\frac{e^{-r^2 (1+2 \sin \theta \cos \theta \cos \alpha)}}{(1+2 \sin \theta \cos \theta \cos \alpha)}\right]_0^\infty d\theta.$$

Now $e^{-\infty}=\frac{1}{e^\infty}=0$ and $e^0=1$.

$$\therefore \quad I=-\frac{1}{2}\int_0^{\pi/2} \left(0-\frac{1}{1+2 \sin \theta \cos \theta \cos \alpha}\right) d\theta.$$

Divide above and below by $\cos^2 \theta$.

$$\therefore \quad I=\tfrac{1}{2} \int_0^{\pi/2} \frac{\sec^2 \theta\, d\theta}{(1+\tan^2 \theta)+2 \tan \theta \cos \alpha}.$$

Put $\tan \theta=t$, $\therefore$ $\sec^2 \theta\, d\theta=dt$.

$$\therefore \quad I=\frac{1}{2}\int_0^{\infty} \frac{dt}{t^2+2t\cos\alpha+\cos^2\alpha+(1-\cos^2\alpha)}$$
$$=\frac{1}{2}\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$
$$=\frac{1}{2}\cdot\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\right]_0^{\infty}$$
$$=\frac{1}{2\sin\alpha}[\tan^{-1}\infty-\tan^{-1}\cot\alpha]$$
$$=\frac{1}{2\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$
$$=\frac{1}{2\sin\alpha}\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\alpha\right)\right]=\frac{\alpha}{2\sin\alpha}.$$

Ex. 35. *Show that*

$$\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\,dx\,dy=\int_0^{\infty}\int_0^{\infty} e^{-r^2}\,r\,dr\,\frac{dt}{1+t^2},$$

where $x^2+y^2=r^2$ and $y=tx$, and hence evaluate the integral.

(Vikram 62)

Put $x=r\cos\theta,\ y=r\sin\theta$; $\therefore\ x^2+y^2=r^2$ and $\tan\theta=\dfrac{y}{x}$

or $y=xt$ where $t=\tan\theta$.

Also $\theta=\tan^{-1}t$; $\therefore\ d\theta=\dfrac{1}{1+t^2}\,dt$.

Also when $\theta=\dfrac{\pi}{2}$, $t=\infty$ and when $\theta=0$, $t=0$.

$$I=\int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\,r\,d\theta\,dr=\int_0^{\infty}\int_0^{\infty} e^{-r^2}\,r\,\frac{dt}{1+t^2}\,dr$$
$$=\int_0^{\pi/2}-\frac{1}{2}\left[e^{-r^2}\right]_0^{\infty}d\theta=\int_0^{\pi/2}-\frac{1}{2}\,[0-1]\,d\theta=\int_0^{\pi/2}\frac{1}{2}\,d\theta$$
$$=\frac{1}{2}\left[\theta\right]_0^{\pi/2}=\frac{\pi}{4}.$$

Ex. 36. *Show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}\,(x^2+y^2+b^2)^{1/2}}=\frac{2\pi}{a+b}$$

ter transforming the integral into r [illegible]

Here the limits of x and y both vary from $-\infty$ to $+\infty$ so that the region of integration extends to whole of x-y plane and hence in polar the limits of integration for r vary from $-\infty$ to $+\infty$ and those of θ from 0 to π thus covering the whole region. Also $dx\,dy = r\,d\theta\,dr$ and $x^2+y^2=r^2$.

Hence after transformation

$$I=\int_0^{\pi}\int_{-\infty}^{\infty}\frac{a.r\,d\theta\,dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}$$

$$=4\int_0^{\pi/2}\int_0^{\infty}\frac{ar\,d\theta\,dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}.$$

Now $$I'=\int_0^{\infty}\frac{r\,dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}$$

$$=\int_0^{\infty}\frac{r\,dr}{(r^2+a^2)\sqrt{[(r^2+a^2)\,(r^2+a^2+b^2-a^2)]}}.$$

Put $r^2+a^2=\dfrac{1}{u}$, $\therefore\ 2r\,dr=-\dfrac{1}{u^2}\,du.$

Also $u=\dfrac{1}{r^2+a^2}$ and when $r=\infty$, $u=0$ and when $r=0$, then $u=\dfrac{1}{a^2}$.

$$I'=\int_{1/a^2}^{0}\frac{-\frac{1}{2u^2}\,du}{\frac{1}{u}\sqrt{\left[\frac{1}{u}\left(\frac{1}{u}+b^2-a^2\right)\right]}}=\frac{1}{2}\int_0^{1/a^2}\frac{du}{\sqrt{[1+u\,(b^2-a^2)]}}$$

$$=\frac{1}{2\,(b^2-a^2)}\left[2\sqrt{\{1+u\,(b^2-a^2)\}}\right]_0^{1/a^2}$$

$$=\frac{1}{(b^2-a^2)}\left(\frac{b}{a}-1\right)=\frac{1}{a\,(a+b)}.$$

$$\because\ I=4\int_0^{\pi/2}a\,d\theta\,I'=4\int_0^{\pi/2}a\,\frac{1}{a\,(a+b)}\,d\theta$$

$$\therefore \quad I=\frac{1}{2}\int_0^\infty \frac{dt}{t^2+2t\cos\alpha+\cos^2\alpha+(1-\cos^2\alpha)}$$
$$=\frac{1}{2}\int_0^\infty \frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$
$$=\frac{1}{2}\cdot\frac{1}{\sin\alpha}\left[\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\right]_0^\infty$$
$$=\frac{1}{2\sin\alpha}[\tan^{-1}\infty-\tan^{-1}\cot\alpha]$$
$$=\frac{1}{2\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\tan\left(\frac{\pi}{2}-\alpha\right)\right]$$
$$=\frac{1}{2\sin\alpha}\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\alpha\right)\right]=\frac{\alpha}{2\sin\alpha}.$$

Ex. 35. *Show that*

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy=\int_0^\infty\int_0^\infty e^{-r^2}\,r\,dr\,\frac{dt}{1+t^2},$$

where $x^2+y^2=r^2$ *and* $y=tx$, *and hence evaluate the integral.*

(Vikram 62)

Put $x=r\cos\theta,\ y=r\sin\theta$; $\therefore\ x^2+y^2=r^2$ and $\tan\theta=\dfrac{y}{x}$

or $y=xt$ where $t=\tan\theta$.

Also $\theta=\tan^{-1}t$; $\therefore\ d\theta=\dfrac{1}{1+t^2}\,dt$.

Also when $\theta=\dfrac{\pi}{2}$, $t=\infty$ and when $\theta=0$, $t=0$.

$$I=\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,d\theta\,dr=\int_0^\infty\int_0^\infty e^{-r^2}\,r\,\frac{dt}{1+t^2}\,dr$$
$$=\int_0^{\pi/2}-\frac{1}{2}\left[e^{-r^2}\right]_0^\infty d\theta=\int_0^{\pi/2}-\frac{1}{2}\,[0-1]\,d\theta=\int_0^{\pi/2}\frac{1}{2}\,d\theta$$
$$=\frac{1}{2}\left[\theta\right]_0^{\pi/2}=\frac{\pi}{4}.$$

Ex. 36. *Show that*

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}\,(x^2+y^2+b^2)^{1/2}}=\frac{2\pi}{a+b}$$

ter transforming the integral into polar,

Here the limits of x and y both vary from $-\infty$ to $+\infty$ so that the region of integration extends to whole of x-y plane and hence in polar the limits of integration for r vary from $-\infty$ to $+\infty$ and those of θ from 0 to π thus covering the whole region Also $dx\ dy = r\ d\theta\ dr$ and $x^2+y^2=r^2$.

Hence after transformation

$$I=\int_0^{\pi}\int_{-\infty}^{\infty} \frac{a\,.\,r\ d\theta\ dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}$$

$$=4\int_0^{\pi/2}\int_0^{\infty} \frac{ar\ d\theta\ dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}.$$

Now $$I'=\int_0^{\infty} \frac{r\ dr}{(r^2+a^2)^{3/2}\sqrt{(r^2+b^2)}}$$

$$=\int_0^{\infty} \frac{r\ dr}{(r^2+a^2)\sqrt{[(r^2+a^2)\ (r^2+a^2+b^2-a^2)]}}.$$

Put $r^2+a^2=\frac{1}{u}$, $\therefore$ $2r\ dr=-\frac{1}{u^2}\ du.$

Also $u=\frac{1}{r^2+a^2}$ and when $r=\infty$, $u=0$ and when $r=0$, then $u=\frac{1}{a^2}$.

$$I'=\int_{1/a^2}^{0} \frac{-\frac{1}{2u^2}\ du}{\frac{1}{u}\sqrt{\left[\frac{1}{u}\left(\frac{1}{u}+b^2-a^2\right)\right]}}=\frac{1}{2}\int_0^{1/a^2} \frac{du}{\sqrt{[1+u\ (b^2-a^2)]}}$$

$$=\frac{1}{2\ (b^2-a^2)}\left[2\sqrt{\{1+u\ (b^2-a^2)\}}\right]_0^{1/a^2}$$

$$=\frac{1}{(b^2-a^2)}\left(\frac{b}{a}-1\right)=\frac{1}{a\ (a+b)}.$$

$$\because\ \ I=4\int_0^{\pi/2} a\ d\theta\ I'=4\int_0^{\pi/2} a\ \frac{1}{a\ (a+b)}\ d\theta$$

$$=\frac{4}{(a+b)}\left[\theta\right]_0^{\pi/2}=\frac{2\pi}{a+b}.$$

Ex. 37. *Show that*

$$\int_0^\infty \int_0^\infty \phi\,(a^2x^2+b^2y^2)\,dx\,dy=\frac{\pi}{4ab}\int_0^\infty \phi(t)\,dt.$$

(Vikram 65)

Let us first put $ax=X$ and $by=Y$

$$\therefore \quad dx\,dy=\frac{1}{ab}\,dX\,dY.$$

$$\therefore \quad I=\frac{1}{ab}\int_0^\infty \int_0^\infty \phi\,(X^2+Y^2)\,dX\,dY.$$

Now put $X=r\cos\theta$, $Y=r\sin\theta$.

$$\therefore \quad I=\frac{1}{ab}\int_0^{\pi/2}\int_0^\infty \phi(r^2)\,r\,d\theta\,dr.$$

Since the limits are constant, we can change the order.

i.e. $$I=\frac{1}{ab}\int_0^\infty \int_0^{\pi/2}\phi(r^2)\,r\,dr\,d\theta=\frac{1}{ab}\int_0^\infty \phi(r^2)\,r\,dr\left[\theta\right]_0^{\pi/2}$$

$$=\frac{\pi}{2ab}\int_0^\infty \phi(r^2)\,r\,dr.$$ Now put $r^2=t$, $\therefore\ 2r\,dr=dt$.

$$\therefore \quad I=\frac{\pi}{4ab}\int_0^\infty \phi(t)\,dt.$$ Proved.

Ex. 38. *Transform the integral*

$$\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)}\,x^{2m-1}y^{2n-1}\,dx\,dy$$

to polar co-ordinates, and deduce that

$$\int_0^{\pi/2}\frac{\cos^{2m-1}\theta\,\sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta=\frac{B(m,n)}{2a^mb^n}=\frac{1}{2a^mb^n}\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}.$$

(Agra 1949)

Putting $x=r\cos\theta$, $y=r\sin\theta$, we get $dx\,dy=r\,d\theta\,dr$ and the limits of r become 0 to ∞ and those of θ, 0 to $\pi/2$ as before.

$$\therefore \quad I=\int_0^{\pi/2}\int_0^\infty e^{-r^2(a\cos^2\theta+b\sin^2\theta)} \times r^{2m-1}\, r^{2n-1}\cos^{2m-1}\theta\sin^{2n-1}\theta\; r\,d\theta\,dr$$

$$=\tfrac{1}{2}\int_0^{\pi/2}\int_0^\infty e^{-t(a\cos^2\theta+b\sin^2\theta)}\, t^{m+n-1}\,dt\,\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta$$

where $r^2=t$.

Now we know that while integrating w.r t. t, θ is to be regarded as constant and also we know from § 3 P, 81 that

$$\int_0^\infty e^{-kt}.t^{n-1}\,dt=\frac{\Gamma n}{k^n}.$$

$$\therefore \quad I=\tfrac{1}{2}\int_0^{\pi/2}\frac{\Gamma(m+n)}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta. \quad \ldots(1)$$

Again given integral can be written as

$$I=\int_0^\infty\int_0^\infty e^{-ax^2}.x^{2m-2}.x\,dx.e^{-by^2}.y^{2n-2}\;y\,dy.$$

Putting $x^2=X$ and $y^2=Y$, we get

$$I=\int_0^\infty e^{-aX}.X^{m-1}.\tfrac{1}{2}\,dX.\int_0^\infty e^{-bY}\;Y^{n-1}.\tfrac{1}{2}\,dY$$

$$=\tfrac{1}{2}.\frac{\Gamma m}{a^m}.\tfrac{1}{2}.\frac{\Gamma n}{b^n} \text{ by the same rule.} \quad \ldots(2)$$

Now (1) and (2) are the values of the same integral and hence they should be equal.

$$\therefore \quad \tfrac{1}{2}\Gamma(m+n)\int_0^{\pi/2}\frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta=\tfrac{1}{4}\frac{\Gamma m\,\Gamma n}{a^m.b^n}.$$

$$\therefore \quad \int_0^{\pi/2}\frac{\cos^{2m-1}\theta\sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta=\frac{1}{2a^mb^n}.\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$=\frac{1}{2a^mb^n}.B(m,n).$$

Ex. 39. *Transform the integral*

$$\int_0^\infty\int_0^\infty x^{l-1}y^{m-1}.e^{-cx-by}\,dx\,dy,$$

by the substitution $x=uv$ and $y=u(1-v)$ and then find the value of integral $\int_0^\infty \frac{v^{l-1}.(1-v)^{m-1}}{\{b(1-v)+av\}^{l+m}}\,dv.$

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}=\begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}=-uv-u(1-v)=-u.$$

$\therefore\quad dx\,dy=-u\,du\,dv=u\,du\,dv$ numerically.

Clearly limits of u and v both are from 0 to ∞.

Also $\quad x^{l-1}y^{m-1}=u^{l-1}v^{l-1}.u^{m-1}(1-v)^{m-1}$

$$=u^{l+m-2}v^{l-1}(1-v)^{m-1}.$$

$$\therefore\quad I=\int_0^\infty\int_0^\infty u^{l+m-2}.v^{l-1}(1-v)^{m-1}.e^{-\{av+b(1-v)\}u}\,u.du\,dv$$

$$=\iint_0^\infty u^{l+m-1}e^{-\{av+b(1-v)\}u}\,v^{l-1}(1-v)^{m-1}\,dv\,du.$$

We have to change $du\,dv$ to $dv\,du$ and thereby we will integrate w.r.t. u, first treating v as constant and, we know that

$$\int_0^\infty e^{-kx}.x^{n-1}\,dx=\frac{\Gamma n}{k^n}. \qquad (\S\ 3\ \text{P. } 81)$$

$$\therefore\quad I=\int_0^\infty \frac{\Gamma(l+m)}{[av+b(1-v)]^{l+m}}.v^{l-1}(1-v)^{m-1}.dv. \qquad \ldots(1)$$

Again the given integral

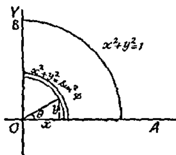
$$=\int_0^\infty e^{-ax}.x^{l-1}.dx\int_0^\infty e^{-by}y^{m-1}\,dy=\frac{\Gamma l}{a^l}\cdot\frac{\Gamma m}{b^m}. \qquad \ldots(2)$$

Since (1) and (2) are the values of the same integral,

$$\therefore\quad \int_0^\infty \frac{v^{l-1}(1-v)^{m-1}}{\{av+b(1-v)\}^{l+m}}\,dv=\frac{1}{\Gamma(l+m)}\cdot\frac{\Gamma l.\Gamma m}{a^l.b^m}=\frac{B(l,m)}{a^l.b^m}.$$

Ex. 40. *Transform* $\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\left(\frac{\sin\phi}{\sin\theta}\right)}\,d\phi\,d\theta$ *by the substitution* $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$ *and show that its value is* π. **(Agra 62 ; Rajputana 65, 63)**

From the given relation, $x^2+y^2=\sin^2\phi$, also $y/x=\tan\theta$. Limits of θ are from 0 to $\pi/2$, *i.e.* it extends to +ive quadrant. Also limits of ϕ are 0 to $\pi/2$ and $\sin \pi/2=1$ and hence it extends to the circle $x^2+y^2=1$ in the +ive quadrant.



Now we want to change the integral in terms of x and y.

$$\frac{\partial(x, y)}{\partial(\phi, \theta)}=\begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{dx}{d\theta} \\ \frac{\partial y}{\partial \phi} & \frac{dy}{d\theta} \end{vmatrix}=\begin{vmatrix} \cos\phi\cos\theta & -\sin\phi\sin\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix}=\cos\phi\sin\phi$$

$$\therefore \quad dx\, dy=\frac{\partial(x, y)}{\partial(\phi, \theta)}.d\phi\, d\theta=\cos\phi\sin\phi\, d\phi\, d\theta.$$

$$\therefore \quad \frac{d\phi}{d\theta}=\frac{1}{\cos\phi\sin\phi}\,dx\,dy.$$

$$\therefore \quad I=\iint\sqrt{\left(\frac{\sin\phi}{\sin\theta}\right)}\cdot\frac{1}{\cos\phi\sin\phi}\,dx\,dy$$

$$=\iint\frac{1}{\sqrt{(1-\sin^2\phi)}}\cdot\frac{1}{\sqrt{(\sin\phi\sin\theta)}}\,dx\,dy$$

$$=\iint\frac{1}{\sqrt{(1-x^2-y^2)}}\cdot\frac{1}{\sqrt{y}}.dx\,dy.$$

Here we shall first integrate w.r.t. y treating x as constant. The above integral extends to +ive quadrant of circle $x^2+y^2=1$. It will be convenient if we change the order *i.e.* $dy\,dx$ so that we have to integrate w.r.t. x treating y as constant and the limits of x in terms of y being $x=0$ to $x=\sqrt{(1-y^2)}$ and those of y are clearly 0 to 1.

$$\therefore \quad I=\int_0^1 \int_0^{\sqrt{(1-y^2)}} \frac{1}{\sqrt{(1-y^2-x^2)}} \cdot \frac{1}{\sqrt{y}}\, dy\, dx$$

$$=\int_0^1 \frac{1}{\sqrt{y}} \left[\sin^{-1} \frac{x}{\sqrt{(1-y^2)}}\right]_0^{\sqrt{(1-y^2)}} dx$$

$$=\int_0^1 \frac{1}{\sqrt{y}} \left[\frac{\pi}{2}-0\right] dx=\frac{\pi}{2}\left[2\sqrt{y}\right]_0^1=\pi.$$

Ex. 40. *Show that*

$$\int_0^c \int_0^{c-x} V\, dx\, dy=\int_0^1 \int_0^c V'u\, dv\, du.$$

If $x+y=u$ and $y=uv$,

$$x=u\ (1-v),\ \ y=uv$$

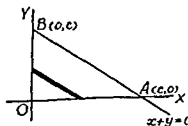
$$dx\, dy=u\, du\, dv \quad \text{or} \quad u\, dv\, du \qquad \text{as in Ex. 39.}$$

Let V be transformed to V' after changing the variables.

From the limits of integration we conclude that the region, of integration is bounded by $y=0$, $y=c-x$, *i.e.* $x+y=c$ and $x=0$, $x=c$.

$u=x+y=0+0=0$ for lower limits when $x=0$, $y=0$,

$=x+y=c$ for upper limit.



Again $v=\frac{y}{u}=\frac{y}{y+x}=\frac{c-x}{c}=0$ for $A\ (c,\ 0)$

$$=\frac{c}{c+0}=1 \text{ for } B\ (0,\ c).$$

$$\therefore \quad I=\int_0^1 \int_0^c V'u\, dv\, du.$$

Ex. 41. *By using the tra[...]* $v=u$, $y=uv$,

prove that $I=\iint \{xy\ (1-x-y)\}$[...] *r the area*

of the triangle bounded by the lines $x=0$, $y=0$, $x+y=1$ is

$$\frac{2\pi}{105}.$$

$$y=uv,\ x+y=u,\quad \therefore\quad x=u-y=u\,(1-v).$$

$$\therefore\quad \sqrt{[xy\,(1-x-y)]}=\sqrt{[u\,(1-v)\,uv\,(1-u)]}$$
$$=uv^{1/2}\sqrt{[(1-u)\,(1-v)]}.$$

Hence as in Q. 40,

$$I=\int_0^1\int_0^1 uv^{1/2}\sqrt{[(1-u)\,(1-v)]}\,u\,du\,dv. \qquad \ldots(1)$$

The limits are adjusted as in Q. 40 as $c=1$

$$\text{or}\quad I=\int_0^1 u^{3-1}\,(1-u)^{\frac{3}{2}-1}\,du\int_0^1 v^{\frac{3}{2}-1}\,(1-v)^{\frac{3}{2}-1}\,dv$$
$$=B(3,\tfrac{3}{2}).B(\tfrac{3}{2},\tfrac{1}{2})$$
$$=\frac{\Gamma 3.\Gamma\frac{3}{2}}{\Gamma(3+\frac{3}{2})}.\frac{\Gamma\frac{3}{2}\ \Gamma\frac{1}{2}}{\Gamma(\frac{3}{2}+\frac{1}{2})}=\frac{2\,!.\frac{1}{2}\Gamma\frac{1}{2}.\frac{1}{2}\Gamma\frac{1}{2}.\Gamma\frac{1}{2}}{\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\Gamma\frac{1}{2}.1}=\frac{2\pi}{105},\ \because\ \Gamma\frac{1}{2}=\sqrt{\pi}.$$

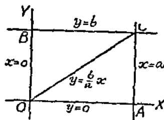
Alternative. We could also put $u=\sin^2\theta$ and $v=\sin^2\phi$ in (1).

$$\therefore\quad I=\int_0^{\pi/2}\int_0^{\pi/2}\sin^4\theta\cos\theta.(2\sin\theta\cos\theta\,d\theta)$$
$$\times\sin\phi.\cos\phi.(2\sin\phi\cos\phi\,d\phi)$$
$$=2\int_0^{\pi/2}\sin^5\theta\cos^2\theta\,d\theta.2\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi$$
$$=2.\frac{4.2.1}{7.5.3}.2.\frac{1.1}{4.2}.\frac{\pi}{2}=\frac{2\pi}{105}.$$

Ex. 42. *Transform the integral* $\int_0^a\int_0^b V\,dx\,dy$ *by the substitution* $x+y=u$, $y=uv$.

$$dx\,dy=u\,dv\,du,$$
$$x=u\,(1-v),\ y=uv.$$

The region is given by rectangle $x=0$, $y=0$, $x=a$, $y=b$.



$$\text{Ans.}\ \int_0^{b/(b+a)}\int_0^{a/(1-v)} V'u\,dv\,du+\int_{b/(b+a)}^1\int_0^{b/v} V'u\,dv\,du.$$

Ex. 43. *Transform the integral*

$$\iiint (x+y+z)^n \, xyz \, dx \, dy \, dz,$$

taken over the volume bounded by $x=0$. $y=0$, $z=0$ and $x+y+z=1$ by the substitution $x+y+z=u$, $y+z=uv$, $z=uvw$.

(Sagar 62 ; Agra 46, 58)

From Ex. 23, we get

$$z=uvw,\ y=uv-uvw=uv\,(1-w),\ x=u-uv=u\,(1-v)$$

and $$dx\,dy\,dz=u^2v\,du\,dv\,dw.$$

Also $V=(x+y+z)^n.xyz=u^n u\,(1-v)\ uv\,(1-w).uvw.$

The given integral

$$=\int_0^1 \int_0^{1-x} \int_0^{1-(x+y)} (x+y+z)^n \, xyz \, dx \, dy \, dz.$$

Hence the transformed integral

$$=\iiint u^{n+3}v^2w\,(1-v)\,(1-w)\,u^2v\,du\,dv\,dw$$

$$=\iiint u^{n+5}\,(v^3-v^4)\,(w-w^2)\,du\,dv\,dw.$$

Limits. When $x=0,\ y=0,\ z=0$, then $u=x+y+z=0$ and when $x+y+z=1$, $u=1$, so that the limits of u are 0 to 1.

$$w=\frac{z}{uv}=\frac{z}{y+z}=0, \quad \text{when } z=0.$$

Also $y+z=1$, *i.e.* $z=1-y$; $\therefore\ w=1-y$.

$\therefore$ when $y=0$, $w=1$ and $y=1$, $w=0$.

$\therefore$ Limits of w are from 0 to 1.

Again $$v=\frac{uv}{u}=\frac{y+z}{x+y+z}=1 \ \text{ if } x=0.$$

Also $x+y+z=1$; $\therefore\ v=y+z=1$.

$$\therefore\ I=\int_0^1 \int_0^1 \int_0^1 u^{n+5}\,(v^3-v^4)\,(w-w^2)\,du\,dv\,dw$$

$$=\left[\frac{u^{n+6}}{n+6}\right]_0^1 \left[\frac{v^4}{4}-\frac{v^5}{5}\right]_0^1 \left[\frac{w^2}{2}-\frac{w^3}{3}\right]_0^1$$

$$=\frac{1}{n+6}\left(\frac{1}{4}-\frac{1}{5}\right)\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{120\,(n+6)}.$$

Ex. 44. *By means of the substitution $x+y+z=u$, $y+z=uv$, $z=uvw$ or otherwise, prove that the value of*

$$\iiint x^{-1/2}y^{-1/2}z^{-1/2}\,(l-x-y-z)^{1/2}\,dx\,dy\,dz,$$

taken over the volume bounded by $x=0$, $y=0$, $z=0$ and $x+y+z=l$ is $\pi^2/4$.

Proceeding exactly as in Q. 43, the given integral after transformation $z=uvw$, $y=uv\,(1-w)$, $x=u\,(1-v)$ is

$$\int_0^1\int_0^1\int_0^1 [u\,(1-v)]^{-1/2}\,[uv\,(1-w)]^{-1/2}\,(uvw)^{-1/2}\,(1-u)^{1/2} \times u^2v\,du\,dv\,dw$$

$$=\int_0^1\int_0^1\int_0^1 u^{1/2}v^0w^{-1/2}\,(1-u)^{1/2}\,(1-v)^{-1/2}\,(1-w)^{-1/2}\,du\,dv\,dw$$

$$=\int_0^1 u^{\frac{3}{2}-1}\,(1-u)^{\frac{3}{2}-1}\,du\int_0^1 v^{1-1}\,(1-v)^{\frac{1}{2}-1}\,dv \times\int_0^1 w^{\frac{1}{2}-1}\,(1-w)^{\frac{1}{2}-1}\,dw.$$

Now $\int_0^1 x^{l-1}\,(1-x)^{m-1}\,dx=B(l,\,m)=\dfrac{\Gamma l.\Gamma m}{\Gamma(l+m)}$.

$\therefore\quad I=B\left(\frac{3}{2},\frac{3}{2}\right).B\left(1,\frac{1}{2}\right).B\left(\frac{1}{2},\frac{1}{2}\right)$

$$=\frac{\Gamma\frac{3}{2}.\Gamma\frac{3}{2}}{\Gamma(\frac{3}{2}+\frac{3}{2})}.\frac{\Gamma 1.\Gamma\frac{1}{2}}{\Gamma(1+\frac{1}{2})}.\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\,(\frac{1}{2}+\frac{1}{2})}.$$

Now $\Gamma\frac{3}{2}=\frac{1}{2}\Gamma\frac{1}{2}=\frac{1}{2}\sqrt{\pi}$ and $\Gamma 3=2\,!=2.$

$$=\frac{\frac{1}{2}\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}}.\frac{\sqrt{\pi}.\sqrt{\pi}}{1}=\frac{\pi^2}{4}.$$

Proved.

Note. See also Q. 15 P. 144.

Ex. 45. *If $x+y+z=u$, $x+y=uv$, $y=uvw$, prove that*

$$\int_0^\infty \int_0^\infty \int_0^\infty V\,dx\,dy\,dz=\int_0^\infty \int_0^1 \int_0^1 V' u^2 v\,du\,dv\,dw.$$

(Rajputana 57)

As in Ex. 23, we can prove that $dx\,dy\,dz=u^2v\,du\,dv\,dw$ and suppose V is transformed to V'.

From given relations, $z=u-uv$, $x=uv-uvw$, $y=uvw$.

Now if $y=0$, then either $u=0$, $v=0$, $w=0$,

if $x=0$, $uv(1-w)=0$, *i.e* $w=1$

and if $z=0$, $u(1-v)=0$, *i.e.* $v=1$

Thus v varies from 0 to 1 and w also varies from 0 to 1 and u varies from 0 to ∞

$$\therefore \quad I=\int_0^\infty \int_0^1 \int_0^1 V' u^2 v\,du\,dv\,dw.$$

Ex. 46. *Prove that* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}=\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$ *throughout the volume bounded by the co-ordinate planes and the plane $x+y+z=1$.*

z varies from 0 to $1-x-y$,

y varies from 0 to $1-x$

and x varies from 0 to 1.

$$\therefore \quad I=\iint\left[-\frac{1}{2(x+y+z+1)^2}\right]_0^{1-x-y} dx\,dy$$

$$=\int_0^1\int_0^{1-x}\left(\frac{1}{2(x+y+1)^2}-\frac{1}{8}\right)dx\,dy$$

$$=\int_0^1\left[-\frac{1}{2(x+y+1)}-\frac{1}{8}y\right]_0^{1-x} dx$$

$$=\int_0^1\left[\frac{1}{2(x+1)}-\frac{1}{4}-\frac{1-x}{8}\right]dx$$

$$=\left[\frac{1}{2}\log(x+1)-\frac{1}{4}x-\frac{1}{8}x+\frac{1}{8}\cdot\frac{x^2}{2}\right]_0^1$$

$$=\tfrac{1}{2}\log 2-\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{16}=\tfrac{1}{2}\left(\log 2-\tfrac{5}{8}\right).$$

Ex. 47. *Show how the change of order in the integral* $\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx \, dy$ *leads to the integration of*

$$\int_0^\infty \frac{\sin rx}{x} \, dx$$

$$I=\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dx \, dy=\int_0^\infty \left[\frac{e^{-xy}}{-x} \sin rx\right]_0^\infty dx$$

$$=-\int_0^\infty \left(0-\frac{\sin rx}{x}\right) dx=\int_0^\infty \frac{\sin rx}{x} \, dx. \qquad \ldots(1)$$

Again $\quad I=\int_0^\infty \int_0^\infty e^{-xy} \sin rx \, dy \, dx.$

Now integrate w.r.t. x first, treating y as constant.

$$\int e^{-xy} \sin rx \, dx$$

$$=\int e^{-xy} \frac{(-\cos rx)}{r}+\int \frac{\cos rx}{r} \, (-y) \, e^{-xy} . dx$$

$$=-\frac{e^{-xy}}{r} \cos rx-\frac{y}{r} \int e^{-xy} \cos rx \, dx$$

$$=-\frac{e^{-xy}}{r} \cos rx-\frac{y}{r}\left[e^{-xy} \frac{\sin rx}{r}-\int \frac{\sin rx}{r} \, (-y) \, e^{-xy} \, dx\right]$$

or $\left(1+\frac{y^2}{r^2}\right) \int e^{-xy} \sin rx \, dx=-\frac{e^{-xy}}{r} \cos rx-\frac{y}{r^2} e^{-xy} \sin rx,$

$$\therefore \quad \int e^{-xy} \sin rx \, dx=-\frac{e^{-xy}}{r^2+y^2} \, [r \cos rx+y \sin rx].$$

$$\therefore \quad I=\int_0^\infty -\left[\frac{e^{-xy}}{r^2+y^2} \, \{r \cos rx+y \sin rx\}\right]_0^\infty dy$$

or, $\quad -\int_0^\infty \left(0-1 . \frac{r}{r^2+x^2}\right) dy=\left[r . \frac{1}{r} \tan^{-1} \frac{y}{r}\right]_0^\infty=\frac{\pi}{2}. \qquad \ldots(2)$

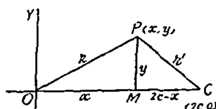
Now (1) and (2) are the values of the same integral.

$$\therefore \quad \int_0^\infty \frac{\sin rx}{x} \, dx=\frac{\pi}{2}.$$

Ex. 48. *If r and r′ be the distances of a point in the plane of reference from two fixed points at a distance 2c apart on the axis of x, then between corresponding limits of integrntion* $\iint 2cy\, dx\, dy = \iint rr'\, dr\, dr'$. **(Agra 33)**

Let O and C be two fixed points on axis of x, such that $OC=2c$ and P be any point (x, y), such that

$OP=r$ and $CP=r'$.



Clearly $PM=y$,

$OM=x$ and $MC=2c-x$.

$\therefore \quad r^2=y^2+x^2$ and $r'^2=y^2+(2c-x)^2$.

$$\frac{\partial(r, r')}{\partial(x, y)}=\begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial r'}{\partial x} & \dfrac{\partial r'}{\partial y} \end{vmatrix}=\begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -\dfrac{(2c-x)}{r'} & \dfrac{y}{r'} \end{vmatrix}$$

$$=\frac{y}{rr'}(x+2c-x)=\frac{2cy}{rr'}.$$

$\therefore \quad dr\, dr'=\dfrac{2cy}{rr'}\, dx\, dy$, $\therefore \quad 2cy\, dx\, dy=rr'\, dr\, dr'$.

$$\therefore \quad \iint 2cy\, dx\, dy=\iint rr'\, dr\, dr'.$$

Ex. 49. *Transform* $\iiint (x-y)(y-z)(z-x)\, dx\, dy\, dz$ *into one in which u, v, w are the independent variables, where*

$$u^3=xyz,\ \frac{1}{v}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \quad \text{and} \quad w^2=x^2+y^2+z^2.$$

(Rajputana 56)

$$\frac{\partial u}{\partial x}=\frac{yz}{3u^2} \text{ etc.}, -\frac{1}{v^2}\frac{\partial v}{\partial x}=-\frac{1}{x^2}\ ;\ \therefore\ \frac{\partial v}{\partial x}=\frac{v^2}{x^2},\ \frac{\partial w}{\partial x}=\frac{x}{w}.$$

$$\Delta=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} \frac{yz}{3u^2} & \frac{zx}{3u^2} & \frac{xy}{3u^2} \\ \frac{v^2}{x^2} & \frac{v^2}{y^2} & \frac{v^2}{z^2} \\ \frac{x}{w} & \frac{y}{w} & \frac{z}{w} \end{vmatrix}=\frac{v^2}{3u^2w}\begin{vmatrix} yz & zx & xy \\ \frac{1}{x^2} & \frac{1}{y^2} & \frac{1}{z^2} \\ x & y & z \end{vmatrix}.$$

Multiply c_1, c_2, c_3 by x^2, y^2, z^2 and hence divide Δ by $x^2y^2z^2$ and take xyz from row no. 1 of new determinant.

$$\therefore\quad \Delta=\frac{v^2}{3u^2w}\frac{xyz}{x^2y^2z^2}\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}.$$

Let $\Delta'=\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}=-\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$ Apply $c_1-c_2,\ c_2-c_3$.

$$=-\begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$=-(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}$$

$$=-(x-y)(y-z)[(y^2+yz+z^2)-(x^2+xy+y^2)]$$

$$=-(x-y)(y-z)[(z-x)+(z^2-x^2)]$$

$$=(x-y)(y-z)(z-x)(x+y+z).$$

$$\therefore \ \Delta = -\frac{v^2}{3u^2wxyz}(x-y)(y-z)(z-x)(x+y+z) = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Now $xyz = u^3$ and $(x+y+z)^2 = \Sigma x^2 + 2\Sigma xy = w^2 + 2\dfrac{u^3}{v}$.

$$\therefore \ \Delta = -\frac{v^2}{3u^2wu^3}(x-y)(y-z)(z-x) \cdot \sqrt{\left(w^2 + \frac{2u^3}{v}\right)}.$$

$$\therefore \ du\,dv\,dw = \frac{\partial(u, v, w)}{\partial(x, y, z)}\,dx\,dy\,dz$$

$$= -\frac{v^2}{3u^5w}\sqrt{\left(w^2 + \frac{2u^3}{v}\right)}$$

$$\times (x-y)(y-z)(z-x)\,dx\,dy\,dz.$$

Hence given integral $= -3\displaystyle\iiint \frac{u^5w}{v}\left(w^2 + \frac{2u^3}{v}\right)^{-1/2} du\,dv\,dw$.

We may reject the $-$ive sign.

50. *Find the area in the* $+$*ive quadrant enclosed between the four curves* $a^2y = x^3$, $b^2y = x^3$, $c^2x = y^3$, $d^2x = y^3$.

Area $= \displaystyle\iint dx\,dy$.

Let the equation of the curves be

$u^2y = x^3$, where u varies from a to b

and $v^2x = y^3$, where v varies from c to d.

Solving, we get $u^2y = \left(\dfrac{y^3}{v^2}\right)^3$ or $u^2v^6 = y^8$. $\therefore \ y = u^{1/4}v^{3/4}$

and similarly $x = u^{3/4}v^{1/4}$.

Now $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{3}{4}u^{-1/4}v^{1/4} & \frac{1}{4}u^{3/4}v^{-3/4} \\ \frac{1}{4}u^{-3/4}v^{3/4} & \frac{3}{4}u^{1/4}v^{-1/4} \end{vmatrix} = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$.

$$\therefore \ dx\,dy = \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv = \tfrac{1}{2}\,du\,dv.$$

$$\therefore \quad \iint dx\, dy = \int_a^b \int_c^d \tfrac{1}{2}\, du\, dv = \tfrac{1}{2}\,(b-a)\,(d-c)$$

$$= \tfrac{1}{2}\,(a-b)\,(c-d).$$

51. *Four parabolas* $y^2=4ax$, $y^2=4bx$, $x^2=4cy$ *and* $x^2=4dy$ *intersect and form a quadrilateral space. Find the area of the space thus enclosed.*

$$\text{Area} = \iint dx\, dy.$$

Let the equation of the curves be

y^2-4ux where u varies from a to b

and $x^2=4vy$ where v varies from c to d.

Solving, we get $\left(\dfrac{x^2}{4v}\right)^2=4ux.$

$$\therefore \quad x=4u^{1/3}v^{2/3} \text{ and } y=4u^{2/3}v^{1/3}.$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{4}{3}u^{-2/3}v^{2/3} & \frac{8}{3}u^{1/3}v^{-1/3} \\ \frac{8}{3}u^{-1/3}v^{1/3} & \frac{4}{3}u^{2/3}v^{-2/3} \end{vmatrix} = \frac{16}{9}-\frac{64}{9}=-\frac{16}{3}.$$

$$\therefore \quad dx\, dy = \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv = -\tfrac{16}{3}\, du\, dv.$$

$$\therefore \quad \iint dx\, dy = \int_a^b \int_c^d \tfrac{16}{3}\, du\, dv = \tfrac{16}{3}\,(b-a)\,(d-c).$$

52. *In case of orthogonal transformations,*

$\xi=l_1x+m_1y+n_1z,\ \eta=l_2x+m_2z+n_2z,\ \zeta=l_3x+m_3y+n_3z,$ *where* $\xi^2+\eta^2+\zeta^2=x^2+y^2+z^2$, *prove that*

$$\iiint f(ax+by+cz)\, dx\, dy\, dz = \iiint f(k\xi)\, d\xi\, dn\, d\zeta$$

where $k=\sqrt{(a^2+b^2+c^2)}$ *and the region of integration in each case is a sphere of radius unity with centre as origin of co-ordinates.*

Let $a=kl_1,\ b=km_1,\ c=kn_1$; $\therefore \quad \sqrt{(a^2+b^2+c^2)}=k.$

$$\therefore \quad ax+by+cz=k\,(l_1x+m_1y+n_1z)=k\xi,$$

$$\frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)}=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=1$$

from Co-ordinate Solid Geometry [see P. 233].

$$\therefore \quad d\xi\, d\eta\, d\zeta=dx\, dy\, dz.$$

$$\therefore \quad \iiint f(ax+by+cz)\; dx\, dy\, dz=\iiint f(k\xi)\, d\xi\, d\eta\, d\zeta,$$

where $\xi^2+\eta^2+\zeta^2=1$.

CHAPTER VI

FOURIER'S SERIES

§ 1. Some important results to be used in this chapter.

1. $\int_{-a}^{a} f(x)\, dx=0$ if $f(x)$ is an odd function of x,

i.e. $$f(-x)=-f(x).$$

Thus $$\int_{-\pi}^{\pi} \sin nx \, dx=0,$$

$\because$ $\sin nx$ is an odd function of x.

$\int_{-a}^{a} f(x)\, dx=2\int_{0}^{a} f(x)\, dx$ if $f(x)$ is an even function of x, *i.e.* $$f(-x)=f(x).$$

Thus $$\int_{-a}^{a} \cos nx\, dx=2\int_{0}^{a} \cos nx\, dx,$$

$\because$ $\cos nx$ is an even function of x.

2. $\therefore$ $\int_{-\pi}^{\pi} \cos nx\, dx=2\int_{0}^{\pi} \cos nx\, dx=2\left[\frac{\sin nx}{n}\right]_{0}^{\pi}=0$

$\because$ $\sin n\pi=0$ and $\sin \pi=0$.

3. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx=2\int_{0}^{\pi} \cos mx \cos nx\, dx.$

$\therefore$ Here the function is an even function of x

$$=\int_{0}^{\pi} [\cos (m+n)\, x+\cos (m-n)\, x]\, dx$$

$$=\left[\frac{\sin (m+n)\, x}{m+n}+\frac{\sin (m-n)\, x}{m-n}\right]_{0}^{\pi}=[0+0]=0.$$

4. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx=2\int_{0}^{\pi} \sin mx \sin nx\, dx,$

$\because \quad \sin(-mx)\sin(-nx)=(-\sin mx)(-\sin nx)$

$=\sin mx \sin nx.$

i e. an even function

$$=\int_0^\pi [\cos(m-n)x-\cos(m+n)x]\,dx=0 \text{ as in (3).}$$

Thus remember that **integral of cos mx, sin mx, cos mx, cos nx and sin mx sin nx within the limits $-\pi$ to π is zero. Above also holds good if the limits are 0 to 2π, where m and n are integers.**

5. $$\int_{-\pi}^{\pi}\cos^2 nx\,dx=2\int_0^\pi \cos^2 nx\,dx=\int_0^\pi (1-\cos 2nx)dx$$
$$=\left[x+\frac{\sin 2nx}{2n}\right]_0^\pi=\pi.$$
$$\int_{-\pi}^{\pi}\sin^2 nx\,dx=2\int_0^\pi \sin^2 nx\,dx=\int_0^\pi (1+\cos 2nx)\,dx$$
$$=\left[x-\frac{\sin 2nx}{2n}\right]_0^\pi=\pi.$$

Thus integral of $\cos^2$ nx and $\sin^2$ nx within the limits $-\pi$ to π is π and if the limits be 0 to π, then the value of integral is $\pi/2$ and if the limits be 0 to 2π, then the value will be 2π.

§ 2. Fourier's Expansion. Expansion of a function of x in a series of sines and cosines of multiples of x.

(Sagar 63 ; Burdwan Hons. 64 ; Karnatak 64 ; Agra 64 ; *Gauhati Hons. 65)*

Let us assume that a given function of x defined in the interval $(-\pi, \pi)$ can be expanded in a trigonometrical series of the form given below :—

$$f(x)=a_0+a_1\cos x+a_2\cos 2x+\ldots+a_n\cos nx+\ldots$$
$$+b_1\sin x+b_2\sin 2x+\ldots+b_n\sin nx+\ldots,$$

where $a_0, a_1, a_2\ldots$ and $b_1, b_2\ldots$ are all constants.

$$f(x)=a_0+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx. \qquad \ldots(1)$$

The above expansion of $f(x)$ is called Fourier's Series for $f(x)$ in the interval $(-\pi, \pi)$.

Let us further assume that the series (1) can be integrated term by term between the limits $-\pi$ to π. The various coefficients are determined as shown below.

To find a_0. We integrate both sides of (1) between the limits $-\pi$ to π.

$$\therefore \quad \int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} \cos nx\,dx \right\} + \sum_{n=1}^{\infty} \left\{ b_n \int_{-\pi}^{\pi} \sin nx\,dx \right\}$$

$$= a_0 \left[x \right]_{-\pi}^{\pi} + 0 + 0 \text{ by (1) and (2) of § 1}$$

$$= 2\pi a_0.$$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v)\,dv. \qquad \ldots(2)$$

To find a_n. Multiply both sides of (1) by $\cos nx$ and then integrate within the limits $-\pi$ to π; we get

$$\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= a_0 \int_{-\pi}^{\pi} \cos nx\,dx + \sum_{n=1}^{\infty} \left\{ c_n \int_{-\pi}^{\pi} \cos^2 nx\,dx \right\} + \sum_{n=1}^{\infty} \left\{ \tfrac{1}{2} b_n \int_{-\pi}^{\pi} \sin 2nx\,dx \right\}$$

$$= 0 + a_n \cdot \pi + 0 \text{ by (1) and (2) of § 1.}$$

$$\therefore \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(v) \cos nv\,dv. \qquad \ldots(3)$$

To find b_n. Multiply both sides of (1) by $\sin nx$ and then integrate within the limits $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x) \sin nx\,dx = a_0 \int_{-\pi}^{\pi} \sin nx\,dx + \sum_{n=1}^{\infty} \left\{ \frac{a_n}{2} \int_{-\pi}^{\pi} \sin 2nx\,dx \right\} + \sum_{n=1}^{\infty} \left[b_n \int_{-\pi}^{\pi} \sin^2 nx\,dx \right]$$

$$=0+0+b_n.\pi \text{ by (1) and (2) of § 1.}$$

$$\therefore \quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} f(v) \sin nv\, dv. \qquad \ldots(4)$$

On putting the values of a_0, a_n and b_n from (2), (3), (4) *in* (1), we get

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)\, dv+\frac{1}{\pi}\sum_{n=1}^{\infty} \cos nx \int_{-\pi}^{\pi} f(v) \cos nv\, dv$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty} \sin nx \int_{-\pi}^{\pi} f(v) \sin nv\, dv \ldots(5)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)\, dv+\frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(v) \cos n\,(v-x)\, dv, \quad \ldots(6)$$

$$-\pi \leqslant x \leqslant \pi.$$

The forms (5) or (6) are called Fourier's expansions.

§ 3. Assumptions for the expansion in a Fourier's series.
(Gujrat 59 ; Agra 61)

1. The given function is assumed to be single-valued, continuous and integrable in the given range.

2. The series $a_0+\sum_{1}^{\infty}(a_n \cos nx+b_n \sin nx)$ is integrable *term by term.* In other words, it means that the series should be uniformly convergent in the interval $-\pi$ to $+\pi$.

3. The series converges to $f(x)$ at every point in $(-\pi, \pi)$, where $f(x)$ is continuous. At ordinary points of discontinuity in the interval $(-\pi, \pi)$, the series converges to $\frac{1}{2}[f(x+0)+f(x-0)]$. At $x=\pm\pi$ the series converges to $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$, when these limits exist.

§ 4. Before doing questions the following integrals should be remembered :—

(1) $$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)}(a \sin bx - b \cos bx)$$
$$= \frac{e^{ax}}{r} \sin (bx - \alpha).$$

(2) $$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{(a^2+b^2)}(a \cos bx + b \sin bx)$$
$$= \frac{e^{ax}}{r} \cos (bx - \alpha),$$

where $r = \sqrt{(a^2+b^2)}$ and $\tan \alpha = \frac{b}{a}$.

The method of successive integration by parts should be noted.

(3) $$\int x^3 \sin 2x\, dx = x^3 \left(-\frac{\cos 2x}{2}\right) - (3x^2)\left(-\frac{\sin 2x}{4}\right)$$
$$+ (6x)\left(\frac{\cos 2x}{8}\right) - 6\left(\frac{\sin 2x}{16}\right).$$

Here we have integrated $\sin 2x$ successively and written the integrations within brackets. x^3 was kept as it is in the beginning and then it was successively differentiated, the various differentiations were written within brackets. The signs were alternately +ive and −ive.

Working Rule.

In the interval $-\pi$ to π,

$$f(x) = a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx),$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

After finding the values of a_0, a_n, b_n and putting

$n=1, 2, 3, \ldots,$ we can find the values of $a_1, b_1, a_2, b_2, \ldots,$ so that

$$f(x)=a_0+(a_1 \cos x+a_2 \cos 2x+a_3 \cos 3x+\ldots)$$
$$+(b_1 \sin x+b_2 \sin 2x+b_3 \sin 3x+\ldots).$$

Ex. 1. *Find a series of sines and cosines of multiples of x, which will represent $\frac{\pi}{2 \sinh \pi} e^x$ in interval $-\pi < x < \pi$.*

(Agra 51, 62 ; Gujrat 59 ; Poona 60)

What is the sum of the series for $x=\pm\pi$.

Let $$f(x)=a_0+\sum_1^{\infty} (a_n \cos nx+b_n \sin nx). \quad \ldots(1)$$

The given interval is $(-\pi, \pi)$ and hence as explained in § 2,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\pi}{2 \sinh \pi} e^x\, dx$$
$$=\frac{1}{2\pi}\cdot\frac{\pi}{2 \sinh \pi}\Big[e^x\Big]_{-\pi}^{\pi}=\frac{1}{4 \sinh \pi} [e^{\pi}-e^{-\pi}]$$
$$=\frac{1}{4 \sinh \pi} (2 \sinh \pi)=\frac{1}{2}. \quad \ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx=\frac{1}{\pi}\cdot\frac{\pi}{2 \sinh \pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx$$
$$=\frac{1}{2 \sinh \pi}\left[\frac{e^x}{1+n^2} (\cos nx+n \sin nx)\right]_{-\pi}^{\pi} \quad (\S\ 4\cdot 3)$$
$$=\frac{1}{2 \sinh \pi}\cdot\frac{1}{(1+n^2)} [e^{\pi} (-1)^n-e^{-\pi} (-1)^n],$$
$$\because \cos n\pi=(-1)^n, \sin n\pi=0$$
$$=\frac{(-1)^n}{(1+n^2)}\cdot\frac{1}{2 \sinh \pi}.2 \sinh \pi=\frac{(-1)^n}{1+n^2}. \quad \ldots(3)$$

$\because$ $e^x-e^{-x}=2 \sinh x$ and $e^x+e^{-x}=2 \cosh x$.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx=\frac{1}{\pi}\cdot\frac{\pi}{2 \sinh \pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx$$

$$=\frac{1}{2\sinh\pi}\left[\frac{e^x}{1+n^2}(\sin nx-n\cos nx)\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2\sinh\pi}\cdot\frac{1}{1+n^2}(-n).(-1)^n(e^\pi-e^{-\pi})=-(-1)^n\frac{n}{n^2+1}$$

$$\ldots(4)$$

Putting for a_0, a_n, b_n from (2), (3) and (4) in (1), we get

$$f(x)=\frac{\pi e^x}{2\sinh\pi}=\frac{1}{2}+\sum_1^\infty\frac{(-1)^n}{n^2+1}(\cos nx-n\sin nx)$$

$$=\tfrac{1}{2}-\tfrac{1}{2}(\cos x-\sin x)+\tfrac{1}{5}(\cos 2x-2\sin 2x)-\ldots$$

Also from § 3, we know that at $x=\pm\pi$, the series converges to

$$\frac{1}{2}[f(-\pi+0)+f(\pi+0)]=\frac{1}{2}\cdot\frac{\pi}{2\sinh\pi}[e^{-\pi}+e^{\pi}].$$

$$=\frac{1}{2}\cdot\frac{\pi}{2\sinh\pi}.2\cosh\pi=\frac{\pi}{2}\coth\pi.$$

Ex. 2. *Find a Fourier's series for $f(x)$ in the interval $(-\pi, \pi)$ where* $f(x)=\pi+x$, $-\pi < x < 0$,

$=\pi-x$, $0 < x < \pi$. **(Agra 45, 68)**

Let $f(x)=a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$. $\ldots(1)$

The given interval is $(-\pi, \pi)$ and hence as explained in § 2,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}f(x)\,dx+\int_0^\pi f(x)\,dx\right].$$

Now for the first integral we shall choose $f(x)=\pi+x$, $-\pi < x < 0$ and for the 2nd integral we shall choose $f(x)=\pi-x$, $0 < x < \pi$.

$$\therefore\quad a_0=\frac{1}{2\pi}\left[\int_{-\pi}^{0}(\pi+x)\,dx+\int_0^\pi(\pi-x)\,dx\right]$$

$$=\frac{1}{2\pi}\left[\left(\pi x+\frac{x^2}{2}\right)_{-\pi}^{0}+\left(\pi x-\frac{x^2}{2}\right)_0^\pi\right]$$

or
$$a_0=\frac{1}{2\pi}\left[\pi^2-\frac{\pi^2}{2}+\pi^2-\frac{\pi^2}{2}\right]=\frac{\pi}{2}, \qquad \ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx.$$

Integrating successively by parts as explained in § 3,

$$a_n=\frac{1}{\pi}\left[(\pi+x)\frac{\sin nx}{n}-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$
$$+\frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n}-(-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}.$$

Now $\sin n\pi=0$ and $\cos n\pi=(-1)^n$.

$$\therefore \quad a_n=\frac{1}{\pi}\left\{0+\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right\}+\frac{1}{\pi}\left\{0-\frac{(-1)^n}{n^2}+\frac{1}{n^2}\right\}$$
$$=\frac{2}{\pi n^2}\{1-(-1)^n\}$$

If n be even, then $(-1)^n=1$, so that

$$a_n=0, \quad i.e. \quad a_2=a_4=a_6=\ldots=0$$

and if n be odd, then $(-1)^n=-1$.

$$\therefore \quad a_n=\frac{4}{\pi n^2}. \quad \text{Hence } a_1=\frac{4}{\pi}\cdot\frac{1}{1^2},\ a_3=\frac{4}{\pi}\cdot\frac{1}{3^2},\ a_5=\frac{4}{\pi}\cdot\frac{1}{5^2}\cdots$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx$$
$$=\frac{1}{\pi}\left[(\pi+x)\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$
$$+\frac{1}{\pi}\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}$$
$$=\frac{1}{\pi}\left(-\frac{\pi}{n}\right)+\frac{1}{\pi}\left\{0-\pi\left(-\frac{1}{n}\right)\right\}=0.$$

$$\therefore \quad b_1 = b_2 = b_3 = \ldots = 0$$

Putting a_0, a_n, b_n in (1), we get

$$f(x) = a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right).$$

(b) *Expand in a series of sines and cosines of multiples of x the function*

$$f(x) = x - \pi, \text{ when } -\pi < x < 0,$$

$$f(x) = \pi - x, \text{ when } \quad 0 < x < \pi.$$

What is the sum of the series for $x = \pm\pi$ *and* $x = 0$. *Hence show that*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \ldots = \frac{\pi^2}{8}$$ **(Vikram 65 ; Sagar 64, 66)**

Ex. 3. *Obtain a Fourier's series for a function defined as*

$f(x) = \cos x$ *for* $0 \leqslant x \leqslant \pi$.

$f(x) = -\cos x$ *for* $-\pi \leqslant x < 0$.

(Rajputana 49 ; Vikram 63)

Let $f(x) = a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$. $\qquad \ldots(1)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\, dx + \int_0^{\pi} f(x)\, dx\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} -\cos x\, dx + \frac{1}{2\pi}\int_0^{\pi} \cos x\, dx$$

$$= \frac{1}{2\pi}\Big[-\sin x\Big]_{-\pi}^{0} + \frac{1}{2\pi}\Big[\sin x\Big]_0^{\pi} = 0. \qquad \ldots(2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x \cos nx\, dx + \int_0^{\pi} \cos x \cos nx\, dx\right].$$

Now we know that $\int_0^{\pi} \cos mx \cos nx\, dx = 0$ and hence 2nd integral is zero.

Also putting $x=-t$ in the first, it becomes

$$\frac{1}{\pi}\int_{\pi}^{0} \cos t \cos nt \, dt=-\frac{1}{\pi}\int_{0}^{\pi} \cos t \cos nt \, dt=0 \text{ as above.}$$

$$\therefore \quad a_n=0. \qquad (3)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x \sin nx \, dx+\int_{0}^{\pi} \cos x \sin nx \, dx\right]$$

$$=-\frac{1}{2\pi}\int_{-\pi}^{0} [\sin (n+1) x+\sin (n-1) x] \, dx$$

$$+\frac{1}{2\pi}\int_{0}^{\pi} [\sin (n+1) x+\sin (n-1) x] \, dx$$

$$=-\frac{1}{2\pi}\left[-\frac{\cos (n+1) x}{n+1}-\frac{\cos (n-1) x}{n-1}\right]_{-\pi}^{0}$$

$$+\frac{1}{2\pi}\left[-\frac{\cos (n+1) x}{n+1}-\frac{\cos (n-1) x}{n-1}\right]_{0}^{\pi}$$

$$=\frac{1}{2\pi}\left[\left(\frac{1}{n+1}+\frac{1}{n-1}\right)-\left\{\frac{\cos (n+1) \pi}{n+1}+\frac{\cos (n-1) \pi}{n-1}\right\}\right]$$

$$-\frac{1}{2\pi}\left[\left\{\frac{\cos (n+1) \pi}{n+1}+\frac{\cos (n-1) \pi}{n-1}\right\}-\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\right]$$

$$=\frac{2}{2\pi}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]-\frac{2}{2\pi}\left[\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right]$$

$$=\frac{1}{\pi}\,\frac{2n}{n^2-1}-(-1)^{n+1}\cdot\frac{2n}{n^2-1}$$

$$\because \quad (-1)^{n+1}=(-1)^2\,(-1)^{n-1}=(-1)^{n-1}$$

$$=\frac{2n}{\pi\,(n^2-1)}\,[1-(-1)^{n+1}.]$$

Now if n is odd, then $(n+1)$ is even and $(-1)^{n+1}=+1$.

$$\therefore \quad b_n=0.$$

Hence $b_1=b_3=b_5=\ldots=0$, n odd.

If n be even, then $(n+1)$ is odd and $(-1)^{n+1}=-1$.

$$\therefore \quad b_n=\frac{4n}{\pi\,(n^2-1)}$$

or
$$b_n=\frac{4n}{\pi(n-1)(n+1)}.$$

$$\therefore\quad b_2=\frac{4}{\pi}\cdot\frac{2}{1.3},\ b_4=\frac{4}{\pi}\cdot\frac{4}{3.5},\ b_6=\frac{4}{\pi}\cdot\frac{6}{5.7}.$$

Hence putting for a_0, a_n, b_n in (1), we get

$$f(x)=\frac{4}{\pi}\left(\frac{2}{1.3}\sin 2x+\frac{4}{3.5}\sin 4x+\frac{6}{5.7}\sin 6x+\ldots\right)$$

Ex. 4. *Find a series of sines and cosines of multiples of x which will represent $x+x^2$ in the interval $-\pi<x<\pi$. Hence show that*

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$$

(Rajputana 62 ; Burdwan Hons. 64 ; Vikram 64 ; Agra 54 ; Jiwaji 66)

Let $\quad f(x)=a_0+\sum_1^{\infty}(a_n\cos nx+b_n\sin nx).\qquad \ldots(1)$

The given interval is $(-\pi, \pi)$ and hence the values a_0, a_n, b_n are found as below :

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx=\frac{1}{2\pi}.2\int_0^{\pi}x^2\,dx$$

(Prop. V)

$$=\frac{1}{\pi}\cdot\frac{\pi^3}{3}=\tfrac{1}{3}\pi^2\qquad\ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx,\quad\because\ x\cos nx \text{ is an odd function of } x$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[0+\frac{2\pi}{n^2}(-1)^n\right]=\frac{4}{n^2}(-1)^n,\qquad\ldots(3)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,dx,$$

$\because$ $x \sin nx$ is an even and $x^2 \sin nx$ is odd function of x

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=-\frac{2}{\pi}\cdot\frac{\pi}{n}(-1)^n=-\frac{4}{2^n}(-1)^n. \qquad \ldots(4)$$

Putting for a_0, a_n and b_n in (1), we get

$$f(x)=x+x^2=\tfrac{1}{3}\pi^2+4\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos nx}{n^2}-\frac{\sin nx}{2n}\right). \quad \ldots(5)$$

Again from § 3 we know that the series converges to

$$\tfrac{1}{2}\{f(-\pi+0)+f(\pi-0)\}, \text{ when } x=\pm\pi.$$

Now $f(x)=x+x^2$; $\therefore$ $f(-\pi+0)=-\pi+(-\pi)^2=-\pi+\pi^2$

and $\qquad f(\pi-0)=\pi+\pi^2$

$\therefore$ Sum of the series $=\frac{1}{2}[-\pi+\pi^2+\pi+\pi^2]=\pi^2$.

Hence we find that at $x=\pi$, the sum of the series in R.H.S. of (5) in π^2. Hence putting $x=\pi$ in (5), we get

$$\pi^2=\frac{\pi^2}{3}+4\sum_{1}^{\infty}(-1)^n\left[\frac{(-1)^n}{n^2}-0\right]$$

or $$\pi^2-\frac{\pi^2}{3}=4\sum_{1}^{\infty}\cdot\frac{1}{n^2} \qquad \because \ (-1)^{2n}=1$$

or $$\frac{2\pi^2}{3}=4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

or $$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

Ex. 5. *Obtain an expansion in a mixed series of sines and cosines of multiples of x which is zero between $-\pi$ and 0 and is equal to e^x beween zero and π and gives its value at three limits.*

Here $\qquad f(x)=0$ for $-\pi<x<0$,

$\qquad f(x)=e^x$ for $\quad 0<x<\pi$.

Let $f(x)=a_0+\Sigma\,(a_n \cos nx+b_n \sin nx)$,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{0}f(x)\,dx+\frac{1}{2\pi}\int_{0}^{\pi}f(x)\,dx$$

$$=0+\frac{1}{2\pi}\int_0^\pi e^x\,dx=\frac{1}{2\pi}(e^\pi-1). \quad \ldots(1)$$

$$a_n=\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx+\frac{1}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$=0+\frac{1}{\pi}\int_0^\pi e^x\cos nx\,dx=\frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^x(\cos nx+n\sin nx)\right]_0^\pi$$

$$=\frac{1}{\pi(1+n^2)}(e^\pi\cos n\pi-1). \quad \ldots(2)$$

Similarly, $b_n=0+\frac{1}{\pi}\int_0^\pi e^x\sin nx\,dx$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx-n\cos nx)\right]_0^\pi$$

$$=-\frac{n}{\pi(1+n^2)}(e^\pi\cos n\pi-1). \quad \ldots(3)$$

$$\therefore\ f(x)=\frac{1}{2\pi}(e^\pi-1)+\frac{1}{\pi}\sum_1^\infty\frac{(e^\pi\cos n\pi-1)}{1+n^2}(\cos nx-n\sin nx).$$

Ex. 6. *Find a Fourier's Series for the function defined by the equations*

$$f(x)=\begin{cases}-1 \text{ for } -\pi<x<0,\\ \ \ 0 \text{ for } \quad x=0,\\ +1 \text{ for } \quad 0<x<\pi.\end{cases}$$

Hence prove that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots$

(Karnatak 64 ; Agra 52)

Proceeding as usual,

$$f(x)=a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx).$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx+\frac{1}{2\pi}\int_0^\pi f(x)\,dx$$

$$=\frac{1}{2\pi}\left[\int^0 -1\,dx+\int_0^\pi 1\,dx\right]=\frac{1}{2\pi}[-\pi+\pi]=0,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$=\frac{1}{\pi}\int_{-\pi}^{0} -1\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx\,dx=0,$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$
$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} -1\sin nx\,dx+\int_{0}^{\pi} 1.\sin nx\,dx\right]$$
$$=\frac{1}{\pi}\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0}-\left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right]=\frac{2}{n\pi}(1-\cos n\pi).$$

When n is even, $\cos n\pi=1$, so that $b_n=0$,

$$i.e.\ b_2=b_4=b_6=0 \quad .=0.$$

When n is odd, then $\cos n\pi=(-1)^n=-1$.

$$\therefore\quad b_n=\frac{4}{n\pi};\quad \therefore\quad b_1=\frac{4}{\pi},\ b_3=\frac{4}{3\pi},\ b_5=\frac{4}{5\pi}.\ .$$

$$\therefore\quad f(x)=\frac{4}{\pi}[\sin x+\tfrac{1}{3}\sin 3x+\tfrac{1}{5}\sin 5x+\ldots].$$

When $x=\frac{\pi}{2}$, *i.e.* $0<x<\pi$, then $f(x)=1$

Hence putting $x=\frac{\pi}{2}$ and $f(x)=1$, we get

$$1=\frac{4}{\pi}[1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots];\quad \therefore\quad \frac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+.\ .$$

Ex. 7. *Find a series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi<x<\pi$ when* $f(x)=\begin{cases}0, & -\pi<x\leqslant 0\\ \frac{1}{4}\pi x, & 0<x<\pi\end{cases}$
and hence deduce that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

(Agra 67 ; Sagar 63)

Let $f(x)=a_0+\sum_{1}^{\infty}(a_n\cos nx+b_n\sin nx)$. ...(1)

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx+\int_{0}^{\pi} f(x)\,dx\right]$$

$$=\frac{1}{2\pi}\left[0+\int_{0}^{\pi} \tfrac{1}{4}\pi x\,dx\right]=\frac{1}{8}\cdot\frac{\pi^2}{2}=\frac{\pi^2}{16}. \qquad \ldots(2)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx+\int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[0+\int_{0}^{\pi}\frac{\pi x}{4}\cos nx\,dx\right]$$

$$=\frac{1}{4}\left[x\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{1}{4n^2}[\cos n\pi-1]=\frac{1}{4n^2}[(-1)^n-1]$$

If n be even, then $(-1)^n=1$

$\therefore\ a_n=0$, i.e., $a_2=a_4=a_6=\ldots=0.$

If n be odd, then $(-1)^n=-1$

$\therefore\ a_n=-\frac{1}{2n^2}$, i.e., $a_1=-\frac{1}{2}$, $a_3=-\frac{1}{2}\cdot\frac{1}{3^2}\ldots$

Similary we can show that $b_n=-\frac{(-1)^n}{4n}\pi$,

so that $\qquad b_1=\frac{\pi}{4},\ b_2=-\frac{\pi}{4.2},\ b_3=\frac{\pi}{4.3}\ldots$

$\therefore\ f(x)=a_0+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x)$
$+(a_3\cos 3x+b_3\sin 3x)+\ldots$

$$f(x)=\frac{\pi^2}{16}+\left(-\frac{1}{2}\cos x+\frac{\pi}{4}\sin x\right)-\frac{\pi}{4.2}\sin 2x$$

$$+\left(-\frac{1}{2.3^2}\cos 3x+\frac{\pi}{4.3}\sin 3x\right)+\ldots$$

Again the series converges to $\frac{1}{2}[f(-\pi+0)+f(\pi-0)]$ when $x=\pi$

$\therefore$ Sum of the series $=\frac{1}{2}\left[0+\frac{\pi}{4}\cdot\pi\right]=\frac{\pi^2}{8}$ at $x=\pi$.

Hence putting $x=\pi$ in the above series, we get

$$\frac{\pi^2}{8}=\frac{\pi^2}{16}+\left(\frac{1}{2}+\frac{1}{2.3^2}+\frac{1}{2.5^2}+\dots\right)$$

or
$$\frac{\pi^2}{8}-\frac{\pi^2}{16}=\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

or
$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+$$

Ex. 8. *Obtain Fourier's series whose sum is equal to $f(x)$, where*

$$f(x)=0 \qquad -\pi \leqslant x < -\frac{\pi}{2} \qquad f\left(-\frac{\pi}{2}\right)=-\frac{\pi}{4}$$

$$f(x)=x \qquad -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad f\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$$

$$f(x)=0 \qquad \frac{\pi}{2} < x \leqslant \pi.$$

Let $f(x)=a_0+\sum_1^\infty (a_n \cos nx+b_n \sin nx)$ in the interval $(-\pi, \pi)$.

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx=\frac{1}{2\pi}\left\{\int_{-\pi}^{-\pi/2} f(x)\, dx+\int_{-\pi/2}^{\pi/2} f(x)\, dx+\int_{\pi/2}^{\pi} f(x)\, dx\right]$$

$$=\frac{1}{2\pi}\left[0+\int_{-\pi/2}^{\pi/2} x\, dx+0\right]=0 \text{ as } x \text{ is odd function of } x.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx=0+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\cos nx\, dx+0$$

as above

or
$$a_n=0.$$

$\because$ $x \cos nx$ is odd function and hence by Prop. V it is zero.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=0+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx+0$$

as above

$$=\frac{1}{\pi}.2\int_0^{\pi/2} x\sin nx\,dx \quad \because\ x\sin nx \text{ is an even function of } x$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[\frac{\sin n\pi/2}{n^2}-\frac{\pi}{2}\cdot\frac{\cos n\pi/2}{n}\right].$$

$$\therefore\ f(x)=\sum_1^{\infty} b_n\sin nx \quad \because\ a_0=0,\ a_n=0$$

$$=\frac{2}{\pi}\sum_1^{\infty}\left\{\frac{\sin\frac{n\pi}{2}}{n^2}-\frac{\pi\cos\frac{n\pi}{2}}{2n}\right\}.\sin nx.$$

§ 4. Odd and even functions of x.

We have seen in the above examples that at times the values of a_n or b_n come out to be zero. Below we shall show how the series will be purely sine or cosine series depending on the nature of the function being odd or even.

We know $f(x)=a_0+\sum_1^{\infty}(a_n\cos nx+b_n\sin nx)$, $(-\pi, \pi)$,

where
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$
$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Also we know that

$$\int_{-a}^{a} f(x)\,dx=0 \quad \text{when } f(x) \text{ is odd, } i.e.\ f(-x)=-f(x)$$

$$=2\int_0^a f(x) \quad \text{when } f(x) \text{ is even, } i.e.\ f(-x)=f(x).$$

$\therefore$ Sum of the series$=\frac{1}{2}\left[0+\frac{\pi}{4}\cdot\pi\right]=\frac{\pi^2}{8}$ at $x=\pi$.

Hence putting $x=\pi$ in the above series, we get

$$\frac{\pi^2}{8}=\frac{\pi^2}{16}+\left(\frac{1}{2}+\frac{1}{2.3^2}+\frac{1}{2.5^2}+\ldots\right)$$

or
$$\frac{\pi^2}{8}-\frac{\pi^2}{16}=\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)$$

or
$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

Ex. 8. *Obtain Fourier's series whose sum is equal to $f(x)$, where*

$$f(x)=0 \qquad -\pi\leqslant x<-\frac{\pi}{2} \qquad f\left(-\frac{\pi}{2}\right)=-\frac{\pi}{4}$$

$$f(x)=x \qquad -\frac{\pi}{2}<x<\frac{\pi}{2} \qquad f\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$$

$$f(x)=0 \qquad \frac{\pi}{2}<x\leqslant\pi$$

Let $f(x)=a_0+\sum_{1}^{\infty}(a_n\cos nx+b_n\sin nx)$ in the interval $(-\pi,\pi)$.

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{-\pi/2}f(x)\,dx+\int_{-\pi/2}^{\pi/2}f(x)\,dx\right.$$
$$\left.+\int_{\pi/2}^{\pi}f(x)\,dx\right]$$

$$=\frac{1}{2\pi}\left[0+\int_{-\pi/2}^{\pi/2}x\,dx+0\right]=0 \text{ as } x \text{ is odd function of } x.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=0+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}x\cos nx\,dx+0$$

as above

or
$$a_n=0.$$

$\because$ $x\cos nx$ is odd function and hence by Prop. V it is zero.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=0+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\sin nx\,dx+0$$

as above

$$=\frac{1}{\pi}.2\int_0^{\pi/2} x\sin nx\,dx \quad \because\ x\sin nx \text{ is an even function of } x$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[\frac{\sin n\pi/2}{n^2}-\frac{\pi}{2}\ \frac{\cos n\pi/2}{n}\right].$$

$$\therefore\ f(x)=\sum_1^{\infty} b_n \sin nx \quad \because\ a_0=0,\ a_n=0$$

$$=\frac{2}{\pi}\sum_1^{\infty}\left\{\frac{\sin\frac{n\pi}{2}}{n^2}-\frac{\pi\cos\frac{n\pi}{2}}{2n}\right\}.\sin nx.$$

§ 4. Odd and even functions of x.

We have seen in the above examples that at times the values of a_n or b_n come out to be zero. Below we shall show how the series will be purely sine or cosine series depending on the nature of the function being odd or even.

We know $f(x)=a_0+\sum_1^{\infty}(a_n\cos nx+b_n\sin nx)$, $(-\pi,\pi)$,

where
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Also we know that

$$\int_{-a}^{a} f(x)\,dx=0 \quad \text{when } f(x) \text{ is odd, } i.e.\ f(-x)=-f(x)$$

$$=2\int_0^a f(x) \qquad \text{when } f(x) \text{ is even, } i.e.\ f(-x)=f(x).$$

Case I. *f(x) being an odd function of x,*

i.e. $f(-x)=-f(x)$.

then $f(x) \cos nx$ is also odd whereas $f(x) \sin nx$ is an even function.

$$\therefore \quad a_0=0,\ a_n=0,\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

or
$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.$$

Hence $f(x)=\sum_1^{\infty} b_n \sin nx$, where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

or
$$b_n=\frac{2}{\pi}\int_0^{\pi} f(v) \sin nv\, dv.$$

Case II. f(x) being even function of x, *i.e* $f(-x)=f(x)$, then $f(x) \cos nx$ is also even whereas $f(x) \sin nx$ is odd.

$$\therefore \quad b_n=0 \text{ and } a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx=\frac{1}{2\pi}\ 2\int_0^{\pi} f(x)\, dx$$

or
$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\, dx.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx=\frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx,$$

$$\therefore \quad f(x)=a_0+\sum_1^{\infty} a_n \cos nx,$$

where $a_n=\frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx=\frac{2}{\pi}\int_0^{\pi} f(v) \cos nv\, dv.$

Hence we conclude that the Fourier's series for odd functions of x in the interval $(-\pi, \pi)$ consists only of sine series whereas for even function of x it consists only of cosine series.

Hence in future we shall not write

$$f(x)=a_0+\sum_1^{\infty} (a_n \cos nx+b_n \sin nx) \text{ in } (-\pi, \pi).$$

But $$f(x)=\sum_{1}^{\infty} b_n \sin nx \text{ when } f(x) \text{ is odd,}$$

where $$b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

or $$f(x)=a_0+\sum_{1}^{\infty} a_n \cos nx \text{ where } f(x) \text{ is even,}$$

where $a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\, dx$ and $a_n=\frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx.$

Ex. 9. *Prove the following relations for values of x between $-\pi$ and $+\pi$ (m is neither zero nor an integer)* :—

$$\sin mx=\frac{2}{\pi} \sin m\pi \left[\frac{\sin x}{1^2-m^2}-2\frac{\sin 2x}{2^2-m^2}+3\frac{\sin 3x}{3^2-m^2}-\ldots\right].$$

(Agra 64, 50 ; Rajputana 55)

We know that $\sin mx$ is an odd function of x (m is neither zero nor an integer as in that case $\sin mx$ would be zero).

$$\therefore \quad f(x)=\sum_{1}^{\infty} b_n \sin nx \qquad (\S\ 4)$$

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx=\frac{2}{\pi}\int_0^{\pi} \sin mx \sin nx\, dx.$

Students may feel that as proved earlier in § 2, the above integral is zero, but it is not the case here as m is not an integer whereas in that article m and n were assumed to be integral.

$$\therefore \quad b_n=\frac{1}{\pi}\int_0^{\pi} [\cos (m-n)\, x-\cos (m+n)\, x]\, dx$$

$$=\frac{1}{\pi}\left[\frac{\sin (m-n)\, x}{m-n}-\frac{\sin (m+n)\, x}{m+n}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\sin (m-n)\, \pi}{m-n}-\frac{\sin (m+n)\, \pi}{m+n}\right].$$

$$\sin (m\pm n)\, \pi=\sin m\pi \cos n\pi \pm \cos m\pi \sin n\pi.$$

But n is an integer and hence $\sin n\pi = 0$.

$\therefore \quad \sin (m \pm n)\pi = \sin m\pi \cos n\pi = (-1)^n \sin m\pi$.

$$\therefore \quad b_n = \frac{1}{\pi} (-1)^n \sin m\pi \left[\frac{1}{m-n} - \frac{1}{m+n}\right]$$
$$= \frac{(-1)^n}{\pi} \sin m\pi . \frac{2n}{m^2-n^2}.$$

$$\therefore \quad b_1 = \frac{2}{\pi} \sin m\pi \left(\frac{-1}{m^2-1^2}\right),\ b_2 = \frac{2}{\pi} \sin m\pi \left(\frac{2}{m^2-2^2}\right).$$
$$b_3 = \frac{2}{\pi} \sin m\pi \left(\frac{-3}{m^2-3^2}\right) \ldots$$

$$\therefore \quad f(x) = \sin mx = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots$$

$$\therefore \quad f(x) = \sin mx$$
$$= \frac{2}{\pi} \sin m\pi \left[\frac{1}{1^2-m^2} \sin x - \frac{2}{2^2-m^2} \sin 2x + \frac{3}{3^2-m^2} \sin 3x, \ldots\right].$$

Ex. 10. *When x lies between $\pm\pi$ and m is not an integer, prove that*

$$\cos mx = \frac{2}{\pi} \sin m\pi \left[\frac{1}{2m} + \frac{m \cos x}{1^2-m^2} - \frac{m \cos 2x}{2^2-m^2} + \frac{m \cos 3x}{3^2-m^2} \cdots\right].$$

(Rajputana 55)

Here $\cos mx$ is an even function of x and we know that when $f(x)$ is an even function, then

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos nx,$$

where $$a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} \cos mx\, dx = \frac{1}{\pi} \left[\frac{\sin mx}{m}\right]_0^{\pi}$$

$$= \frac{1}{m\pi} \sin m\pi \qquad (m \text{ is not an integer})$$

and $$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos mx \cos nx\, dx$$

$$=\frac{1}{\pi}\int_0^{\pi} \cos (m-n)\, x+\cos (m+n)\, x\, dx$$

$$=\frac{1}{\pi}(-1)^n.\sin m\pi.\frac{2m}{m^2-n^2} \text{ as in above example.}$$

$$\therefore \quad a_1=\frac{2 \sin m\pi}{\pi}\left[\frac{-1}{m^2-1^2}\right],\ a_2=\frac{2 \sin m\pi}{\pi}\left[\frac{1}{m^2-2^2}\right]\cdots$$

$$\therefore \quad f(x)=\frac{2 \sin m\pi}{\pi}\left[\frac{1}{2m}+\frac{m}{1^2-m^2} \cos x-\frac{m}{2^2-m^2} \cos 2x + \frac{m}{3^2-m^2} \cos 3x\ldots\right].$$

(b) *Deduce from above that*

$$\cot u=\frac{1}{u}+\frac{2u}{u^2-\pi^2}+\frac{2u}{u^2-4\pi^2}+\cdots$$

We have proved in part (a) that

$$\cos mx=\frac{2}{\pi} \sin m\pi \left(\frac{1}{2m}+\frac{m \cos x}{1^2-m^2}-\frac{m \cos 2x}{2^2-m^2} + \frac{m \cos 3x}{3^2-m^2}-\cdots\right).$$

Let us put $m\pi=u$ and $x=\pi$.

$$\therefore \quad \cos u=\frac{2}{\pi} \sin u \left\{\frac{\pi}{2u}+\frac{u}{\pi}\frac{\cos \pi}{\left(1^2-\frac{u^2}{\pi^2}\right)}-\frac{u}{\pi}\frac{\cos 2\pi}{\left(2^2-\frac{u^2}{\pi^2}\right)}+\cdots\right\}$$

$$\cot u=\frac{1}{u}+\frac{2u}{u^2-\pi^2}+\frac{2u}{u^2-4\pi^2}\cdots \qquad \text{Proved.}$$

Ex. 11. *Obtain Fourier's series in* $(-\pi, \pi)$ *for* $f(x)=x \cos x$. **(Gujrat 52)**

Here $f(x)=x \cos x$ is odd function of x.

$$\therefore \quad f(x)=\sum_1^{\infty} b_n \sin nx,$$

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx=\frac{1}{\pi}\int_0^{\pi} 2x \cos x \sin nx\, dx$

$$=\frac{1}{\pi}\int_0^\pi x\,[\sin(n+1)x+\sin(n-1)x]\,dx$$

Integrate by parts

$$=\frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}-\frac{\cos(n-1)x}{n-1}\right\}-1\left\{-\frac{\sin(n+1)x}{n+1}-\frac{\sin(n-1)x}{(n-1)}\right\}\right]_0^\pi$$

$$=\frac{1}{\pi}\left[-\pi\left\{\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right\}\right],$$

$\because\ \sin n\pi=0$ and $\cos n\pi=(-1)^n$

$$=-(-1)^{n-1}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]=(-1)^n\frac{2n}{n^2-1}.$$

Putting $n=2, 3, \ldots$, we get $b_2, b_3, \ldots$ but when $n=1$, then $b_1=\infty$ and hence we find b_1 as we found b_n.

$$b_1=\frac{2}{\pi}\int_0^\pi f(x)\sin x\,dx,\text{ putting } n=1$$

$$=\frac{2}{\pi}\int_0^\pi x\cos x\sin x\,dx=\frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$=\frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-(1)\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2}\right]=-\tfrac{1}{2}.$$

$$\therefore\quad f(x)=\sum_1^\infty b_n\sin nx=b_1\sin x+\sum_{n=2}^\infty b_n\sin nx$$

$$=-\tfrac{1}{2}\sin x+\sum_{n=2}^\infty(-1)^n\cdot\frac{2n}{n^2-1}\sin nx.$$

Ex. 12. *Obtain Fourier's series for the expansion of $f(x)=x\sin x$ in the interval $-\pi<x<\pi$.*
Hence deduce that

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\ldots\qquad\text{(Agra 66, 53)}$$

Here $f(x) = x \sin x$ is an even function of x and hence

$$f(x) = a_0 + \sum_{1}^{\infty} a_n \cos nx \, dx,$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx = \frac{1}{\pi}\int_0^{\pi} x \sin x \, dx$

$$= \frac{1}{\pi}\left\{x(-\cos x) - (1)(-\sin x)\right\}_0^{\pi} = \frac{1}{\pi}(\pi) = 1.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\{\sin(n+1)x - \sin(n-1)x\}\, dx$$

$$= \frac{1}{\pi}\Bigg[x\left\{\frac{\cos(n-1)x}{n-1} - \frac{\cos(n+1)x}{n+1}\right\}$$

$$- (1)\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\}_0^{\pi}\Bigg]$$

$$= \frac{1}{\pi}\,\pi\left\{\frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1}\right\} = (-1)^{n-1}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

or $$a_n = (-1)^{n-1} \cdot \frac{2}{(n-1)(n+1)}.$$

As before if we put $n = 1$ and then $a_1 = \infty$ and hence proceed directly putting $n = 1$ in the original value of a_n, we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2} \text{ as in Ex. 11.}$$

Now putting $n = 2, 3 \ldots$ in a_n, we get

$$a_2 = -\frac{2}{1.3},\ a_3 = \frac{2}{2.4},\ a_4 = -\frac{2}{3.5} \ldots$$

$$\therefore\ f(x) = a_0 + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$=1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x+\ldots$$

Ex. 13. *Prove that*

$$\frac{x(\pi^2-x^2)}{12}=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\ldots(-\pi<x<\pi).$$

(Rajputana 53)

Here $\frac{x(\pi^2-x^2)}{12}$ is an odd function of x and hence

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx\, dx,$$

where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx=\frac{2}{\pi}\int_0^{\pi}\frac{x(\pi^2-x^2)}{12}\sin nx\, dx$

$$=\frac{1}{6\pi}\int_0^{\pi}(\pi^2x-x^3)\sin nx\, dx.$$

Integrating successively by parts, we get

$$b_n=\frac{1}{6\pi}\left\{(\pi^2x-x^3)\left(-\frac{\cos nx}{n}\right)-(\pi^2-3x^2)\left(-\frac{\sin nx}{n^2}\right)\right.$$
$$\left.+(-6x)\left(\frac{\cos nx}{n^3}\right)-(-6)\left(\frac{\sin nx}{n^4}\right)\right\}_0^{\pi}$$

$$=\frac{1}{6\pi}\left\{0-0-6\pi\frac{(-1)^n}{n^3}+0\right\}=-\frac{(-1)^n}{n^3}.$$

$\therefore\quad b_1=\frac{1}{1^3},\ b_2=-\frac{1}{2^3},\ b_3=\frac{1}{3^3},\ b_4=-\frac{1}{4^3}\ldots\ldots$

$$\therefore\quad \frac{x(\pi^2-x^2)}{12}=\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\frac{\sin 4x}{4^3}\ldots\infty.$$

Ex. 14. *Prove that for all values of x between $-\pi$ and π,*

$$\frac{x}{2}=\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x\ldots$$

Here again $\frac{x}{2}$ is an odd function ; $\therefore\ f(x)=\sum_{1}^{\infty} b_n \sin nx,$

(Gauhati Hon's 65)

when $b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx=\frac{2}{\pi}\int_0^\pi \frac{x}{2}\sin nx\,dx$

$$=\frac{1}{\pi}\left[\frac{x(-\cos nx)}{n}-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^\pi=-\frac{(-1)^n}{n}\text{ etc.}$$

Ex. 15. *Prove that if* $-\pi<x<\pi$,

$$\frac{\pi\sinh ax}{2\sinh a\pi}=\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}\ldots$$

Here $f(x)$ is an odd function of x

$\therefore\quad f(x)=\sum_1^\infty b_n\sin nx$, where $b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

or $b_n=\frac{2}{\pi}\int_0^\pi\frac{\pi}{2}\cdot\frac{\sinh ax}{\sinh a\pi}\sin nx=\frac{1}{\sinh a\pi}\int_0^\pi\frac{e^{ax}-e^{-ax}}{2}\sin nx\,dx$

$$=\frac{1}{2\sinh a\pi}\int_0^\pi(e^{ax}\sin nx-e^{-ax}\sin nx)\,dx$$

$$=\frac{1}{2\sinh a\pi}\cdot\frac{1}{(a^2+n^2)}\Big[e^{ax}(a\sin nx-n\cos nx)$$

$$-e^{-ax}(-a\sin nx-n\cos nx)\Big]_0^\pi.$$

Put $\sin n\pi=0$ and $\cos n\pi=(-1)^n$.

$$\therefore\quad b_n=\frac{1}{2\sinh a\pi}\cdot\frac{1}{(a^2+n^2)}\left[\{e^{a\pi}(-n)(-1)^n+n\}\right.$$

$$\left.-\{e^{-a\pi}(-n)(-1)^n+n\}\right]$$

$$=\frac{(-n)(-1)^n}{2\sinh a\pi}\cdot\frac{1}{(a^2+n^2)}\cdot(e^{a\pi}-e^{-a\pi})=-\frac{n(-1)^n}{a^2+n^2}.$$

$$\therefore\quad b_1=\frac{1}{a^2+1^2},\ b_2=\frac{-2}{a^2+2^2},\ b_3=\frac{3}{a^2+3^2},\ \ldots$$

$$\therefore\ \frac{\pi}{2}\frac{\sinh ax}{\sinh a\pi}=\frac{1}{a^2+1^2}\sin x-\frac{2}{a^2+2^2}\sin 2x+\frac{3}{a^2+3^2}\sin 3x+\ldots$$

Ex. 16. *Prove that for all values of x between* $-\pi$ *and* π,

$$\frac{\pi}{2}\frac{\cosh ax}{\sinh ax}=\frac{1}{2a}-\frac{a\cos x}{a^2+1^2}+\frac{a\cos 2x}{a^2+2^2}-\frac{a\cos 3x}{a^2+3^2}+\ldots$$

Here $f(x)$ is an even function of x and hence

$$f(x)=a_0+\sum_{1}^{\infty} a_n \cos nx.$$

where a_0 and a_n have values written before, etc

$$a_0=\frac{1}{2a} \text{ and } a_n=(-1)^n.\frac{a}{a^2+n^2}$$

§ 5. Fourier's Series for interval 0 to π.

Let $f(x)$ be a given function of x satisfying the conditions laid down in § 3 in the interval $(0, \pi)$. Then we shall show that the function can be expanded in a series of sines and cosines of multiples of x for any values of x in the above interval.

Sine series : Interval $(0, \pi)$.

Let $f(x)=b_1 \sin x+b_2 \sin 2x+b_3 \sin 3x+\ldots+b_n \sin nx$. (1)

or $$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

In order to find the value of b_n, we multiply both sides of (1) by $\sin nx$ and integrate between 0 to π.

$$\therefore \int_0^{\pi} f(x) \sin nx\, dx=b_n \int_0^{\pi} \sin nx.\sin nx\, dx.$$

$\because \int_0^{\pi} b_1 \sin x \sin nx\, dx$, $\int_0^{\pi} b_2 \sin 2x \sin nx\, dx$ etc. all vanish by § 2.

$$\therefore \int_0^{\pi} f(x) \sin nx\, dx=b_n \int_0^{\pi} \frac{(1-\cos 2nx)}{2}\, dx=b_n.\frac{\pi}{2}.$$

$$\therefore \mathbf{b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.}$$

Note. It may be noted that the value of b_n for interval $(0, \pi)$ is same as the value of b_n for interval $(-\pi, \pi)$ when $f(x)$ was an odd function of x as proved in § 4.

$$\therefore \quad f(x) = \sum_{1}^{\infty} b_n \sin nx$$

$$= \sum_{1}^{\infty} \left\{ \frac{2}{\pi} \int_0^{\pi} f(v) \sin nv \right\} dv \sin nx$$

The sum of the above series is equal to

$$\tfrac{1}{2} [f(x+0) + f(x-0)]$$

at every point x between 0 and π ; it is equal to 0 at $x=0$ and at $x=\pi$.

Cosine series. Let

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx$$

$$= a_0 + \sum_{1}^{\infty} a_n \cos nx. \qquad \ldots(1)$$

In order to find a_0, we integrate both sides between 0 to π.

$$\therefore \quad \int_0^{\pi} f(x)\, dx = a_0 \int_0^{\pi} dx + 0 = a_0 \pi.$$

$$\therefore \quad a_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\, dx = \frac{1}{\pi} \int_0^{\pi} f(v)\, dv. \qquad \ldots(2)$$

In order to find a_n, we multiply both sides of (1) by $\cos nx$ and integrate between the limits 0 to π and remember that $\int_0^{\pi} \cos x\, dx$, $\int_0^{\pi} \cos x \cos nx\, dx$, $\int_0^{\pi} \cos 2x \cos nx\, dx$ all vanish by § 2

$$\therefore \quad \int_0^{\pi} f(x) \cos nx\, dx = a_n \int_0^{\pi} \cos^2 nx\, dx$$

$$= a_n \int_0^{\pi} \left(\frac{1 + \cos 2nx}{2} \right) dx = a_n \cdot \frac{\pi}{2}.$$

$$\therefore \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} f(v) \cos nv\, dv.$$

Note. It may be noted here that the values of a_0 and a_n for interval $(0, \pi)$ are same as *the corresponding values* of a_0 and a_n for interval $(-\pi, \pi)$ when $f(x)$ is an even function of x [§ 4].

$$\therefore \quad f(x)=a_0+\sum_1^{\infty} a_n \cos nx$$
$$=\frac{1}{\pi}\int_0^{\pi} f(v)\,dv+\sum_{n=1}^{\infty}\left\{\frac{2}{\pi}\int_0^{\pi} f(v)\cos v\,dv\right\}\cos nx.$$

The sum of the above series is $\frac{1}{2}$ [f(x+0)+f(x−0)] at any point x in the above interval and it is equal to f(+0) for x=0 and f(π−0) for x=π.

Ex. 17. *Find a series of cosines of muliples of x which will represent log* $\left(2 \sin \frac{x}{2}\right)$ *in the interval* $(0, \pi)$.

(Gauhati Hons. 67)

$$f(x)=a_0+\sum_1^{\infty} a_n \cos nx=a_0+a_1 \cos x+a_2 \cos 2x$$
$$+a_3 \cos 3x+\ldots$$

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi}\left(\log 2+\log \sin \frac{x}{2}\right)dx$$
$$=\frac{1}{\pi}\left\{\pi \log 2+\int_0^{\pi/2} \log \sin t\,(2\,dt)\right\}, \text{ where } \frac{x}{2}=t.$$

$$\therefore \quad \log 2+\frac{2}{\pi}\cdot\frac{\pi}{2}\log\frac{1}{2} \text{ (Ex. 1 P. 4)}=\log 2-\log 2=0.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\log\left(2 \sin \frac{x}{2}\right)\cos nx\,dx.$$

Put $\frac{x}{2}=t$ and adjust the limits.

$$\therefore \quad a_n=\frac{2}{\pi}\int_0^{\pi/2}(\log 2 \sin t).\cos 2nt\,(2\,dt).$$

integrate by parts

$$-\frac{4}{\pi}\left[\frac{\sin 2nt}{2n}\log(2\sin t)\right.$$
$$\left.-\frac{1}{2n}\int \sin 2nt\ \frac{1}{2\sin t}.2\cos t\ dt\right]$$
$$=*0-\frac{2}{\pi n}\int_0^{\pi/2}\sin 2nt\ \frac{\cos t}{\sin t}\ dt$$
$$=-\frac{1}{\pi n}\int_0^{\pi/2}\left[\frac{\sin(2n+1)\ t}{\sin t}+\frac{\sin(2n-1)\ t}{\sin t}\right]dt.$$

Now we know from author's Integral Calculus, § 56 P. 237, that

$$\int\frac{\sin mx}{\sin x}\,dx=\left[\frac{2\sin(m-1)\ x}{m-1}\right]+\int\frac{\sin(m-2)\ x}{\sin x}\,dx$$
$$\therefore\quad \int_0^{\pi/2}\frac{\sin(2n+1)\ x\ dx}{\sin x}=\left[\frac{2\sin 2nx}{2n}\right]_0^{\pi/2}$$
$$+\int_0^{\pi/2}\frac{\sin(2n-1)\ x}{\sin x}\,dx$$

or
$$I_{2n+1}=0+I_{2n-1}$$
$$\therefore\quad I_{2n-1}=0+I_{2n-3}=0+I_{2n-5}=\ldots=0+I_1$$

where
$$I_1=\int_0^{\pi/2}\frac{\sin t}{\sin t}\,dt=\int_0^{\pi/2}dt=\frac{\pi}{2}.$$

$$\therefore\quad a_n=-\frac{1}{\pi n}\left[I_{2n+1}+I_{2n-1}\right]=-\frac{1}{\pi n}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=-\frac{1}{n}.$$
$$\therefore\quad a_1=-1,\ a_2=-\tfrac{1}{2},\ a_3=-\tfrac{1}{3}.$$
$$\therefore\quad f(x)=\log\left(2\sin\frac{x}{2}\right)=-\cos x-\tfrac{1}{2}\cos 2x-\tfrac{1}{3}\cos 3x\ldots$$

*Note. We have put here $\left[\sin 2nt.\log(2\sin t)\right]_0^{\pi/2}=0.$

For $t=\pi/2$ it vanishes, but for $t=0$ it assumes indeterminate form $0\times\infty$.

$$\underset{t\to 0}{\text{Lt}}\ \frac{\log(2\sin t)}{\operatorname{cosec} 2nt}=\frac{\infty}{\infty}=\frac{\frac{1}{2\sin t}.2\cos t}{-2n\operatorname{cosec} 2nt.\cot 2nt}$$

$$=-\frac{1}{2n}\left(\frac{\sin^2 2nt.\cos t}{\sin t.\cos 2nt}\right)\frac{0}{0}$$

$$=\underset{t\to 0}{\text{Lt}}-\frac{1}{2n}\left[\frac{\sin^2 2nt\,(-\sin t)+\cos t\,(2\sin 2nt\cos 2nt).2n}{\cos t.\cos 2nt-2n\sin 2nt\sin t}\right]$$

$=0.$

Ex. 18. *Prove that when $0<x<\pi$,*

$$\log\operatorname{cosec} x=\log 2+\cos 2x+\tfrac{1}{2}\cos 4x+\ldots+\frac{2}{n}\cos nx\ldots.$$

Now $\log\operatorname{cosec} x=-\log\sin x$.

$$a_0=\frac{1}{\pi}\int_0^\pi-\log\sin x\,dx=-\frac{1}{\pi}.2\int_0^{\pi/2}\log\sin x\,dx$$

$$=-\frac{2}{\pi}.\frac{\pi}{2}\log\tfrac{1}{2}=-\log\tfrac{1}{2}=\log 2.$$

$$a_n=\frac{2}{\pi}\int_0^\pi-\log\sin x.\cos nx\,dx$$

$$=-\frac{2}{\pi}\left[\frac{\sin nx}{n}\log\sin x-\frac{1}{n}\int_0^\pi\sin nx\frac{\cos x}{\sin x}\,dx\right]$$

$$-0+\frac{1}{\pi n}\int_0^\pi\left[\frac{\sin(n+1)x}{\sin x}+\frac{\sin(n-1)x}{\sin x}\right]dx. \quad \ldots(1)$$

But $I_{n+1}=\left[\frac{2\sin nx}{n}\right]_0^\pi+I_{n-1}=I_{n-1}=I_{n-3}=I_{n-5}$

as in last question.

$\therefore$ If n is odd, then $n+1$ etc. are even.

$$\therefore\quad I_{n+1}=I_{n-1}=I_{n-3}\ldots$$

$$=I_2=\int_0^\pi\frac{\sin 2x}{\sin x}\,dx=\int_0^\pi 2\cos x\,dx=0.$$

$$\therefore\quad a_1=a_3=a_5=\ldots=0.$$

If n is even, then $n+1$ etc. are odd.

$$\therefore\quad I_{n+1}=I_{n-1}\ldots=I_1=\int_0^\pi\frac{\sin x}{\sin x}\,dx=\pi.$$

$$\therefore\quad a_n=\frac{1}{\pi n}.[I_{n+1}+I_{n-1}]=\frac{1}{\pi n}[\pi+\pi]=\frac{2}{n}\text{ from (1).}$$

$$\therefore\quad a_2=\tfrac{2}{2},\ a_4=\tfrac{2}{4},\ a_6=\tfrac{2}{6}\ldots$$

$$\therefore\ f(x)=\log 2+\tfrac{2}{2}\cos 2x-\tfrac{2}{4}\cos 4x-\ldots\ \frac{2}{n}\cos nx\ \ldots$$

Ex. 19. *Find function $f(x)$ in terms of sines which shall be equal to x from $x=0$ to $x=a$, equal to a from $x=a$ to $x=\pi-a$ and then equal to $\pi-x$ from $x=\pi-a$ to π.*

(Rajputana 58)

$$f(x)=\sum_{1}^{\infty} b_n \sin nx=b_1 \sin x+b_2 \sin 2x+\ldots$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{a} x\sin nx\,dx+\frac{2}{\pi}\int_a^{\pi-a} a\sin nx\,dx$$

$$+\frac{2}{\pi}\int_{\pi-a}^{\pi}(\pi-x)\ \sin nx\,dx$$

according to the definition of the function

$$=\frac{2}{\pi}[I_1+I_2+I_3].$$

$$I_1=\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^a=\frac{\sin na}{n^2}-\frac{a}{n}\cos na.$$

$$I_2=\left[-a\,\frac{\cos nx}{n}\right]_a^{\pi-a}=-\frac{a}{n}\left[\cos n(\pi-a)-\cos na\right].$$

$$I_3=\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi-a}^{\pi}$$

$$=0-\left\{a\left(-\frac{\cos n(\pi-a)}{n}\right)-\frac{\sin n(\pi-a)}{n^2}\right\}.$$

$$b_n=\frac{2}{\pi}[I_1+I_2+I_3]=\frac{2}{\pi}\left[\frac{\sin na}{n^2}+\frac{\sin(n\pi-na)}{n^2}\right]$$

$$=\frac{2}{\pi}\left[\frac{\sin na}{n^2}-\frac{\sin n\pi\cos na-\cos n\pi\sin na}{n^2}\right]$$

$$=\frac{2}{\pi}\left[\frac{\sin na}{n^2}-(-1)^n.\frac{\sin na}{n^2}\right]$$

$\because\ \sin n\pi=0$ and $\cos n\pi=(-1)^n$.

$$=\frac{2}{\pi}\cdot\frac{\sin na}{n^2}\,[1-(-1)^n].$$

n even. Then $(-1)^n=1$,

$$\therefore\quad b_n=0 \text{ i.e, } b_2=b_4=b_6=\ldots=0.$$

n odd. Then $(-1)^n=-1$, $\therefore\quad b_n=\frac{4}{\pi}\cdot\frac{\sin na}{n^2}$.

$$\therefore\quad b_1=\frac{4}{\pi}\sin a,\ b_3=\frac{4}{\pi}\frac{\sin 3a}{3^2},\ b_5=\frac{4}{\pi}\frac{\sin 5a}{5^2}\ \ldots$$

$$\therefore\quad f(x)=\frac{4}{\pi}\left[\sin a\sin x+\frac{1}{3^2}\sin 3a.\sin 3x\right.$$
$$\left.+\frac{1}{5^2}\sin 5a.\sin 5x+\ldots\right].$$

Ex. 20. *If* $\phi(x)=\frac{x}{2}$ *when* $0<x<\alpha$

$$=\frac{\alpha}{2} \text{ when } \alpha<x<\pi-\alpha$$
$$=\tfrac{1}{2}(\pi-x) \text{ when } \pi-\alpha<x<\pi,$$

prove that $\phi(x)=\frac{2}{x}\left[\sin\alpha\sin x+\frac{1}{9}\sin 3\alpha\sin 3x\right.$

$$\left.+\frac{1}{25}\sin 5\alpha.\sin 5x+\ldots\right].\ \textbf{(Agra 57)}$$

A careful look at the question will show that it is same as Q. 19 except that the function in this question is multiplied by $\frac{1}{2}$ and so is the answer.

Ex. 21. *Find a series of cosines of multiples of* x *which will represent* $f(x)$ *in the interval* $(0,\pi)$ *where*

$$f(x)=0,\qquad 0\leqslant x<\pi/2 \text{ and } f(\pi/2)=\pi/4$$
$$=2/\pi,\ \pi/2<x\leqslant\pi.$$

$$f(x)=a_0+\sum_1^\infty a_n\cos nx.$$

where $$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} f(x)\cos nx\,dx+\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi} f(x)\cos nx\,dx$$

$$=0+\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi}{2}\cos nx\,dx=\left[\frac{\sin nx}{n}\right]_{\frac{1}{2}\pi}^{\pi}=0-\frac{1}{n}\sin\frac{n\pi}{2}.$$

$\therefore\quad a_1=-1,\ a_2=0,\ a_3=\frac{1}{3},\ a_4=0\ a_5=-\frac{1}{5}\ldots$

Also $a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} f(x)\,dx+\frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\pi} f(x)\,dx$

$$=0+\frac{1}{\pi}\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi}{2}\,dx=\frac{1}{2}\left[x\right]_{\frac{1}{2}\pi}^{\pi}=\frac{1}{2}\left\{\pi-\frac{\pi}{2}\right\}=\frac{\pi}{4}.$$

$$\therefore\quad f(x)=a_0+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\ldots$$

$$=\frac{\pi}{4}-[\cos x-\tfrac{1}{3}\cos 3x+\tfrac{1}{5}\cos 5x\ldots].$$

Ex. 22. *Find a series of cosines of multiples of x, when the function $f(x)$ is defined as*

$$f(x)=\begin{cases}\frac{1}{3}\pi & 0\leqslant x<\frac{1}{3}\pi,\\ 0 & \frac{1}{3}\pi<x<\frac{2}{3}\pi,\\ -\frac{1}{3}\pi & \frac{2}{3}\pi<x\leqslant\pi.\end{cases}$$

In the interval $(0,\pi)$,

$$f(x)=a_0+\sum_1^{\infty} a_n\cos nx.$$

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx=\frac{1}{\pi}\left[\int_0^{\frac{1}{3}\pi}\frac{\pi}{3}\,dx+\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}0\,dx+\int_{\frac{2}{3}\pi}^{\pi}-\frac{\pi}{3}\,dx\right]$$

$$=\frac{1}{\pi}\left\{\frac{\pi}{3}\left(\frac{\pi}{3}-0\right)+0-\frac{\pi}{3}\left(\pi-\tfrac{2}{3}\pi\right)\right\}=0.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{\int_0^{\frac{1}{3}\pi}\frac{\pi}{3}\cos nx\,dx+\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi}0\,dx+\int_{\frac{2}{3}\pi}^{\pi}-\frac{\pi}{3}\cos nx\,dx\right\}$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{3}\left[\left\{\frac{\sin nx}{n}\right\}_0^{\frac{1}{3}\pi}-\left\{\frac{\sin nx}{n}\right\}_{\frac{2}{3}\pi}^{\pi}\right]$$

$$=\frac{2}{3n}\left[\sin\frac{n\pi}{3}-\left\{\sin n\pi-\sin\frac{2n\pi}{3}\right\}\right]$$

$$=\frac{2}{3n}\left[\sin\frac{n\pi}{3}+\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{2}{3n}\left[\sin\frac{n\pi}{3}+\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}\right]$$

$$=\frac{2}{3n}\sin\frac{n\pi}{3}\,[1-(-1)^n].$$

$\because \sin n\pi=0$ and $\cos n\pi=(-1)^n$.

If n be even then $(-1)^n=1$ and when n is multiple of three, then $\sin\frac{n\pi}{3}=0$

$\therefore \quad a_n=0$, i.e $a_2=a_4=a_6=0$.

Also $\qquad a_3=a_7=a_9=0$.

If n be odd but not multiple of three, then

$$a_n=\frac{4}{3n}\sin\frac{n\pi}{3}.$$

$$\therefore \quad a_1=\frac{4}{3}\cdot\frac{\sqrt{3}}{2}=\frac{2}{\sqrt{3}},\ a_5=\frac{4}{3.5}\sin\frac{5\pi}{3}=-\frac{4}{3}\cdot\frac{1}{5}\cdot\frac{\sqrt{3}}{2}$$

$$=-\frac{2}{\sqrt{3}}\cdot\frac{1}{5}\text{ and so on.}$$

$$\therefore \quad f(x)=a_0+\Sigma a_n\cos nx$$

$$=\frac{2}{\sqrt{3}}\left[\cos x-\frac{\cos 5x}{5}+\frac{\cos 7x}{7}\ldots\right].$$

Ex. 23. *Find a series of sines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where*

$$f(x)=\tfrac{1}{3}\pi, \qquad 0<x<\tfrac{1}{3}\pi$$
$$=0, \qquad \tfrac{1}{3}\pi<x<\tfrac{2}{3}\pi,$$
$$=-\tfrac{1}{3}\pi, \quad \tfrac{2}{3}\pi<x<\pi.$$

Can you represent this function by a series of cosines of multiples of x as well. Explain your answer. Draw graphs of these series and find the values of sine and cosine series, when $x=-\frac{1}{3}\pi$, $=-\frac{2}{3}\pi$ and $-\pi$. (Agra 59)

$$f(x)=\sum_{1}^{\infty} b_n\sin nx \text{ (sine series)}$$

where $\quad b_n=\dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\, dx$

or $\quad b_n=\dfrac{2}{\pi}\left[\displaystyle\int_0^{\pi/3}\frac{\pi}{3}\sin nx\, dx+\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 0\, dx+\int_{\frac{2}{3}\pi}^{\pi}-\frac{\pi}{3}\sin nx\, dx\right]$

$$=\frac{2}{\pi}\cdot\frac{\pi}{3}\left[\left(\frac{-\cos nx}{n}\right)_0^{\pi/3}-\left(\frac{-\cos nx}{n}\right)_{\frac{2}{3}\pi}^{\pi}\right]$$

$$=\frac{2}{3n}\left(1-\cos\frac{n\pi}{3}+\cos n\pi-\cos\frac{2n\pi}{3}\right)$$

$$=\frac{2}{3n}\left[1+(-1)^n-\left(\cos\frac{n\pi}{3}+\cos\frac{2n\pi}{3}\right)\right]$$

$$=\frac{2}{3n}\left[1+(-1)^n-\cos\frac{n\pi}{3}\{1+(-1)^n\}\right],$$

$$\because\quad \cos\frac{2n\pi}{3}=\cos\left(n\pi-\frac{n\pi}{3}\right)=(-1)^n\cos\frac{n\pi}{3}$$

or $\quad b_n=\dfrac{2}{3n}\,[1+(-1)^n]\,2\sin^2\dfrac{n\pi}{6},\quad \because\ 1-\cos\theta=2\sin^2\dfrac{\theta}{2}.$

Clearly $b_n=0$, when n is odd.

$$\therefore\quad b_1=b_3=b_5=\ldots=0$$

Also when n is even, but a multiple of 6 say $6k$, then

$$\sin\frac{n\pi}{6}=\sin k\pi=0\,,\quad \therefore\quad b_6=b_{12}=\ldots=0.$$

But if n is even, but not a multiple of 6, then b_n is not zero.

$$\therefore\quad b_2=\frac{2}{3.2}\ 2.2\sin^2\frac{\pi}{3}=1,$$

$$b_4=\frac{2}{3.4}.2.2\sin^2\frac{2\pi}{3}=\tfrac{1}{2},$$

$$b_8=\tfrac{1}{4},\ b_{10}=\tfrac{1}{5}\text{ and so on.}$$

$$\therefore\quad f(x)=\sin 2x+\tfrac{1}{2}\sin 4x+\tfrac{1}{4}\sin 8x+\tfrac{1}{5}\sin 10x+\ldots$$

$$=2\left[\tfrac{1}{2}\sin 2x+\tfrac{1}{4}\sin 4x+\tfrac{1}{8}\sin 8x+\tfrac{1}{10}\sin 10x+\ldots\right]$$

Cosine series. We have already expanded the above function in cosine series in Q. 22,

i e., $$f(x)=\frac{2}{\sqrt{3}}\,(\cos x-\tfrac{1}{5}\cos 5x+\tfrac{1}{7}\cos 7x+\ldots).$$

Graph of sine series. Let the series be denoted by $y=f(x)$. For +ive values of x, $0 \leqslant x < \pi$.

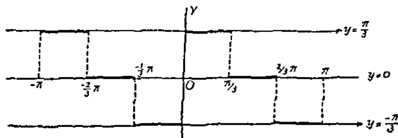
$$\therefore\quad y=\pi/3,\qquad 0 \leqslant x < \tfrac{1}{3}\pi,$$
$$y=0,\qquad \tfrac{1}{3}\pi < x < \tfrac{2}{3}\pi,$$
$$y=-\tfrac{1}{3}\pi,\qquad \tfrac{2}{3}\pi < x < \pi.$$

For −ve values of x, *when* $-\pi < x < 0$, *let us put*

$$x=-z\,;\quad \therefore\quad -\pi < -z < 0$$

or $$\pi > z > 0 \quad\text{or}\quad 0 < z < \pi,$$
$$y=f(x)=f(-z)=-f(z)$$
$$\therefore\quad y=-\tfrac{1}{3}\pi \quad\text{for } -\tfrac{1}{3}\pi < x < 0$$
$$=0 \quad\text{for } -\tfrac{2}{3}\pi < x < -\tfrac{1}{3}\pi$$
$$=\tfrac{1}{3}\pi \quad\text{for } -\pi < x < -\tfrac{2}{3}\pi.$$

Hence the graph of sine series is as under :—



The portion between $-\pi$ and π repeats indefinitely. Here if x be changed into $-x$, then y changes to $-y$, so that there is symmetry in opposite quadrants.

Again we know from § 5 P. 246, that sum of the sine series is equal to $\frac{1}{2}\,[f(x+0)+f(x-0)]$ at every point x between 0 and π and it is equal to zero at $x=0$ and at $x=\pi$.

$$\therefore \quad f\left(-\frac{\pi}{3}\right)=-f\left(\frac{\pi}{3}\right)=-\frac{1}{2}\left[f\left(\frac{\pi}{3}+0\right)+f\left(\frac{\pi}{3}-0\right)\right]$$

$$=-\frac{1}{2}\left[0+\frac{\pi}{3}\right]=-\frac{\pi}{6};$$

$$f\left(-\frac{2\pi}{3}\right)=-f\left(\frac{2\pi}{3}\right)=-\frac{1}{2}\left[f\left(\frac{2\pi}{3}+0\right)+f\left(\frac{2\pi}{3}-0\right)\right]$$

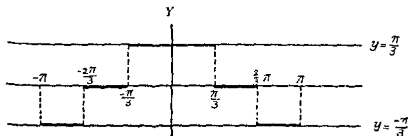
$$=-\frac{1}{2}\left[-\frac{\pi}{3}+0\right]=\frac{\pi}{6}.$$

Also $\qquad f(-\pi)=f(\pi)=0.$

Graph of cosine series.

In the interval 0 to π the graph is same as of sine series and also if x be changed into $-x$, then $\cos(-\theta)$ being equal to $\cos\theta$, y does not change, so that there is symmetry about y-axis.

Hence the graph is as shown below.



Also we know from § 5 P. 244 that the sum of the above series is $\frac{1}{2}[f(x+0)+f(x-0)]$ at any point x in the interval $(0, \pi)$ and is equal to $f(+0)$ for $x=0$ and $f(\pi-0)$ for $x=\pi$.

$$\therefore \quad f\left(-\frac{\pi}{3}\right)=f\left(\frac{\pi}{3}\right)=\frac{1}{2}\left[f\left(\frac{\pi}{3}+0\right)+f\left(\frac{\pi}{3}-0\right)\right]$$

$$=\frac{1}{2}\left[0+\frac{\pi}{3}\right]=\frac{\pi}{6},$$

$$f\left(-\frac{2\pi}{3}\right)=f\left(\frac{2\pi}{3}\right)=\tfrac{1}{2}\left[f\left(\frac{2\pi}{3}+0\right)+f\left(\frac{2\pi}{3}-0\right)\right]$$
$$=\tfrac{1}{2}\left[-\frac{\pi}{3}+0\right]=-\frac{\pi}{6},$$
$$f(-\pi)=f(\pi)=f(\pi-0)=-\tfrac{1}{3}\pi.$$

Ex. 24. *The function $f(x)$ is defined as follows for the interval $(0, \pi)$:*

$$f(x)=\frac{3}{2}x \qquad \text{when } 0 \leqslant x \leqslant \frac{\pi}{3}$$
$$-\frac{\pi}{2} \qquad \text{when } \frac{\pi}{3} < x < \tfrac{2}{3}\pi$$
$$=\tfrac{3}{2}(\pi-x) \text{ when } \tfrac{2}{3}\pi \leqslant x \leqslant \pi.$$

Show that $f(x)=\frac{6}{\pi}\sum_{1}^{\infty}\frac{\sin\frac{1}{3}(2n-1)\pi \sin(2n-1)\pi}{(2n-1)^2}$.

The function is expanded into sine series and we know that in the interval $(0, \pi)$,

$$f(x)=\sum_{1}^{\infty} b_n \sin nx,$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$
$$=\frac{2}{\pi}\left[\int_0^{\pi/3}\tfrac{3}{2}x\sin nx \right.$$
$$\left.+\int_{\pi/3}^{\frac{2}{3}\pi}\tfrac{1}{2}\pi\sin nx\, dx+\int_{\frac{2}{3}\pi}^{\pi}\tfrac{3}{2}(\pi-x)\sin nx\, dx\right]$$
$$=\frac{2}{\pi}\left[\frac{3}{2}\left\{x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/3}-\frac{\pi}{2}\left\{\frac{\cos nx}{n}\right\}_{\pi/3}^{\frac{2}{3}\pi}\right.$$
$$\left.+\frac{3}{2}\left\{(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\frac{2}{3}\pi}^{\pi}\right]$$
$$=\frac{2}{\pi}\left[\frac{3}{2}\left(-\frac{\pi}{3n}\cos\frac{n\pi}{3}+\frac{1}{n^2}\sin\frac{n\pi}{3}\right)-\frac{\pi}{2n}\left(\cos\frac{2}{3}\pi n-\cos\frac{\pi}{3}n\right)\right.$$
$$\left.+\frac{3}{2}\left\{0+(\pi-\tfrac{2}{3}\pi)\frac{1}{n}\cos\frac{2n\pi}{3}+\frac{1}{n^2}\sin\frac{2n\pi}{3}\right\}\right]$$

$$=\frac{2}{\pi}\cdot\frac{3}{2n^2}\left[\sin\frac{n\pi}{3}+\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$=\frac{3}{\pi n^2}\left[\sin\frac{n\pi}{3}+\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}\right]$$

$$=\frac{3}{\pi n^2}\cdot\sin\frac{n\pi}{3}\,[1-(-1)^n].$$

$\therefore\quad b_n=0$, when n is even i e $b_2=b_4=b_6=\ldots=0$

When n is odd say $2k-1$, where $k=1, 2, 3\ldots\infty$, then

$$b_n=\frac{3}{\pi(2k-1)^2}\sin(2k-1)\frac{\pi}{3}(2)=\frac{6}{\pi}\cdot\frac{\sin(2k-1)\pi/3}{(2k-1)^2}.$$

$$\therefore\quad f(x)=\Sigma b_n\sin nx.$$

$$\therefore\quad f(x)=\frac{6}{\pi}\sum_{1}^{\infty}\frac{\sin(2k-1)\pi/3}{(2k-1)^2}\sin(2k-1)x.$$

Ex. 25. *Find a cosine as well as sine series of multiples of x, when function is defined as*

$$f(x)=\begin{cases}0 & 0\leqslant x<\frac{\pi}{2},\\ \frac{\pi}{2} & \frac{\pi}{2}<x\leqslant\pi.\end{cases}$$

We have already found the cosine series in Q. 21 P. 250. Below we find the sine series.

If the function is expanded in sine, series then

$$f(x)=\sum_{1}^{\infty}b_n\sin nx,$$

where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx=0+\frac{2}{\pi}\int_{\pi/2}^{\pi}$

$$=-\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

or $\qquad b_n=-\frac{1}{n}\left\{(-1)^n-\cos\frac{n\pi}{2}\right\}$

$\therefore \quad b_1 = -(-1-0) = 1,\ b_2 = -\frac{1}{2}(1+1) = -1.$

$b_3 = -\frac{1}{3}\{-1-0\} = \frac{1}{3},$

$b_4 = -\frac{1}{4}(1-1) = 0,\ b_5 = -\frac{1}{5}(-1-0) = \frac{1}{5}$ etc.

$\therefore \quad f(x) = \sin x - \sin 2x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x \ldots$

Ex. 26. *If* $f(x) = x$ *for* $0 \leqslant x \leqslant \frac{\pi}{2}$,

$$f(x) = \pi - x \text{ for } \frac{\pi}{2} \leqslant x \leqslant \pi.$$

Expand the above function in a sine series and also in cosine series. **(Agra 42)**

Hence deduce that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Sine series. $f(x) = \sum_{1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + \left(0 + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right)\right]$$

$$\frac{4}{\pi n^2}\sin\frac{n\pi}{2}.$$

If n be even, then $b_n = 0$ *i.e.* $b_2 = b_4 = \ldots = 0$

If n be odd, then $b_1 = \frac{4}{\pi}.1,\ b_3 = \frac{4}{\pi.3^2}(-1),\ b_5 = \frac{4}{\pi.5^2}(1).$

$$\therefore \quad f(x)=\frac{4}{\pi}\left[\sin x-\frac{1}{3^2}\sin 3x+\frac{1}{5^2}\sin 5x \ldots\right]$$

Deduction. If we put $x=\frac{\pi}{2}$, then the sum of the series is

$$\frac{1}{2}\left[f\left(\frac{\pi}{2}+0\right)+f\left(\frac{\pi}{2}-0\right)\right]=\frac{1}{2}\left[\frac{\pi}{2}+\left(\pi-\frac{\pi}{2}\right)\right]=\frac{\pi}{2}.$$

$$\therefore \quad \frac{\pi}{2}=\frac{4}{\pi}\left[\sin\frac{\pi}{2}-\frac{1}{3^2}\sin\frac{3\pi}{2}+\frac{1}{5^2}\sin\frac{5\pi}{2}\ldots\right]$$

or

$$\frac{\pi^2}{8}=\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right].$$

Cosine series. $f(x)=a_0+\sum_1^\infty a_n \cos nx$.

$$a_0=\frac{1}{\pi}\int_0^\pi f(x)\,dx=\frac{1}{\pi}\left[\int_0^{\frac{1}{2}\pi} x\,dx+\int_{\frac{1}{2}\pi}^{\pi}(\pi-x)\,dx\right]=\frac{\pi}{4}.$$

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\int_0^{\frac{1}{2}\pi} x\cos nx\,dx+\int_{\frac{1}{2}\pi}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$=\frac{2}{\pi}\left[\left\{x.\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\frac{1}{2}\pi}+\left\{(\pi-x)\frac{\sin nx}{n}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\frac{1}{2}\pi}^{\pi}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2}+\frac{1}{n^2}\left(\cos\frac{n\pi}{2}-1\right)+\left(0-\frac{\pi}{2n}\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right]$$

$$=\frac{2}{\pi}.\frac{1}{n^2}\left[2\cos\frac{n\pi}{2}-\{1+(-1)^n\}\right].$$

When n is odd, then $\cos\frac{n\pi}{2}=0$ and $(-1)^n=-1$,

$$\therefore \quad a_n=0, \ \ i.e, \ \ a_1=a_3=\ldots=0,$$

Again when n is even but multiple of 4 say $4k$, then

$\cos \frac{n\pi}{2}=\cos 2k\pi=1$; $\therefore$ $a_n=0$, *i.e.* $a_4=a_8=\ldots=0$

Again when n – even but not multiple of four, then

$$a_n=\frac{2}{\pi n^2}[-2-2]=-\frac{8}{\pi n^2}.$$

$$\therefore \quad a_2=-\frac{8}{\pi}\cdot\frac{1}{2^2},\ a_6=-\frac{8}{\pi}\cdot\frac{1}{6^2},\ a_{10}=-\frac{8}{\pi}\ \frac{1}{10^2}\cdots$$

$$\therefore \quad f(x)=\frac{\pi}{4}-\frac{8}{\pi}\left[\frac{\cos 2x}{2^2}+\frac{\cos 6x}{6^2}+\frac{\cos 10x}{10^2}+\ldots\right].$$

Ex. 27. *Find a series of sines of multiples of x which will represent $f(x)$ in the interval $(0, \pi)$, where $f(x)=0$, $0\leqslant x<\pi/2, f(\pi/2)=\pi/4, f(x)=\pi/2, \pi/2<x<\pi. f(\pi)=0$.*

(Rajputana 68)

$$f(x)=\sum_1^\infty b_n \sin nx, \text{ where } b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$$

or $$b_n=\frac{2}{\pi}\left[\int_0^{\frac{1}{2}\pi} f(x)\sin nx\,dx+\int_{\frac{1}{2}\pi}^{\pi} f(x)\sin nx\,dx\right]$$

$$=\frac{2}{\pi}\left[0+\int_{\frac{1}{2}\pi}^{\pi}\frac{\pi}{2}\sin nx\,dx\right]=-\left[\frac{\cos nx}{n}\right]_{\frac{1}{2}\pi}^{\pi}$$

$$=\frac{1}{n}\left[\cos\frac{n\pi}{2}-\cos n\pi\right].$$

If n is odd, then $\cos n\pi=-1$ and $\cos\frac{n\pi}{2}=0$; $\therefore$ $b_n=\frac{1}{n}$.

$$\therefore \quad b_1=1,\ b_3=\tfrac{1}{3},\ b_5=\tfrac{1}{5}\ldots.$$

If n is even and multiple of 4, then $n/2$ is even and $\cos(2k\pi)=1$; $\therefore$ $b_n=0$, *i.e.* $b_4=b_8=b_{12}=\ldots=0$.

If n is even but not a multiple of 4, then $\cos n\pi=1$ and

$\cos\frac{n\pi}{2}=\cos(\text{odd }\pi)=-1$; $\therefore$ $b_n=\frac{1}{n}(-1-1)=-\frac{2}{n}$.

$\therefore$ $b_2=-\frac{2}{2}$, $b_6=-\frac{2}{6}$, $b_{10}=-\frac{2}{10}$ and so on.

$$\therefore \quad f(x)=b_1 \sin x+b_2 \sin 2x+b_3 \sin 3x+b_4 \sin 4x+\ldots$$
$$=\sin x-\tfrac{2}{2} \sin 2x+\tfrac{1}{3} \sin 3x+\tfrac{1}{5} \sin 5x-\tfrac{2}{6} \sin 6x$$
$$+\tfrac{1}{7} \sin 7x+\tfrac{1}{9} \sin 9x-\tfrac{2}{10} \sin 10x+\ldots$$

Sum of the series when $x=\pi/2$ is

$$\tfrac{1}{2}[f(\pi/2+0)+\tfrac{1}{2}f(\pi/2-0)=\tfrac{1}{2}(\pi/2+0)=\pi/4.$$
$$\therefore \quad \pi/4=1-0+\tfrac{1}{3}(-1)+\tfrac{1}{5}(1) \quad 0+\tfrac{1}{7}(-1)\ldots$$

or $\quad \pi/4=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+$

Ex. 28. *Find a series of sines of multiples of x which will represent f(x) in the interval $(0, \pi)$, where*

$$f(x)=\frac{\pi x}{4} \qquad 0 \leqslant x \leqslant \frac{\pi}{2}$$
$$=\frac{\pi}{4}(\pi-x), \ \frac{\pi}{2}<x \leqslant \pi.$$

Also find the series of cosines of multiples of x.

Sine series $\quad f(x)=\sum_{1}^{\infty} b_n \sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$
$$=\frac{2}{\pi}\left[\int_0^{\pi/2} f(x) \sin nx \, dx+\int_{\pi/2}^{\pi} f(x) \sin nx \, dx\right]$$
$$=\frac{2}{\pi}\left[\int_0^{\pi/2} \frac{\pi x}{4} \sin nx \, dx+\int_{\pi/2}^{\pi} \frac{\pi}{4}(\pi-x) . \sin nx \, dx\right]$$
$$=\frac{2}{\pi} \cdot \frac{\pi}{4}\left[\left\{x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi/2}\right.$$
$$\left.+\left\{(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$
$$=\frac{1}{2}\left[\left\{-\frac{\pi}{2n} \cos \frac{n\pi}{2}+\frac{1}{n^2} \sin \frac{n\pi}{2}\right\}\right.$$
$$\left.+\left\{0+\frac{\pi}{2n} \cos \frac{n\pi}{2}+\frac{1}{n^2} \sin \frac{n\pi}{2}\right\}\right]$$
$$=\frac{1}{n^2} \sin \frac{n\pi}{2}.$$

If n is even, then $b_n=0$, *i.e.* $b_2=b_4=b_6=\ldots=0$.

Also $b_1=\frac{1}{1^2}.1,\ b_3=\frac{1}{3^2}(-1),\ b_5=\frac{1}{5^2}.1.$

$\therefore\ f(x)=\sin x-\frac{1}{3^2}\sin 3x+\frac{1}{5^2}\sin 5x-\frac{1}{7^2}\sin 7x+\ldots$

Graph of sine series.

For +ive values of x, $0 < x < \pi$

$y=\frac{\pi}{4}x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$, $y=\frac{\pi}{4}(\pi-x)$ for $\frac{\pi}{2} \leqslant x \leqslant \pi$.

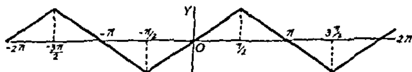
For $-$ive values of x when $-\pi < x < 0$, let us put $x=-z$. $\therefore$ $-\pi < -z < 0$ or $\pi > z > 0$ or $0 < z < \pi$; we have

$$y=f(x)=f(-z)=-f(z)$$

$$\therefore\ y=-\frac{\pi}{4}x \text{ for } -\frac{\pi}{2} \leqslant x \leqslant 0$$

and $$y=-\frac{\pi}{4}(\pi-x) \text{ for } -\pi \leqslant x \leqslant -\frac{\pi}{2}.$$

If x be changed into $-x$, y changes to $-y$ so that there is symmetry in opposite quadrants. The portion between $-\pi$ to π repeats indefinitely on either side.



Cosine series. $f(x)=a_0+\sum_1^\infty a_n \cos nx.$

$$a_0=\frac{1}{\pi}\int_0^\pi f(x)\,dx=\frac{1}{\pi}\left[\int_0^{\pi/2}\frac{\pi x}{4}\,dx+\int_{\pi/2}^{\pi}\frac{\pi}{4}(\pi-x)\,dx\right]$$

$$=\frac{1}{\pi}\cdot\frac{\pi}{4}\left[\left\{\frac{x^2}{2}\right\}_0^{\pi/2}+\left\{\pi x-\frac{x^2}{2}\right\}_{\pi/2}^{\pi}\right]=\frac{\pi^2}{16},$$

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2}\frac{\pi x}{4}\cos nx\,dx+\int_{\pi/2}^{\pi}\frac{\pi(\pi-x)}{4}\cos nx\,dx\right]$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{4}\left[\left\{x\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi/2}\right.$$
$$\left.+\left\{(\pi-x)\frac{\sin nx}{n}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$
$$=\frac{1}{2}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2}+\frac{1}{n^2}\left(\cos\frac{n\pi}{2}-1\right)+\left(0-\frac{\pi}{2n}\sin\frac{n\pi}{2}\right)\right.$$
$$\left.-\left(\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\cos\frac{n\pi}{2}\right)\right]$$
$$=\frac{1}{2}\left[\frac{2}{n^2}\cos\frac{n\pi}{2}-\frac{1}{n^2}\{1+(-1)^n\}\right]$$

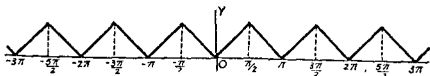
If n is odd, then clearly $a_n=0$, $\therefore\ a_1=a_3=a_5=\ldots=0$

If n is even but multiple of 4, then

$$\cos\frac{n\pi}{2}=\cos(\text{even }\pi)=1$$
$$\therefore\quad a_n=\frac{1}{2}\left[\frac{2}{n^2}-\frac{2}{n^2}\right]=0.$$
$$\therefore\quad a_4=a_8=a_{12}=\ldots=0.$$

If n is even but not multiple of 4 say 2, 6, 10..., then $\frac{n}{2}$ is odd, so that $\cos\frac{n\pi}{2}=\cos(\text{odd }\pi)=-1$.

$$\therefore\quad a_n=\frac{1}{2}\left[-\frac{2}{n^2}-\frac{1}{n^2}(1+1)\right]=-\frac{2}{n^2}.$$
$$\therefore\quad a_2=-\frac{2}{2^2},\ a_6=-\frac{2}{4^2},\ a_6=-\frac{2}{6^2}\ldots$$



$$\therefore\quad f(x)=a_0+a_1\cos x+a_2\cos 2x+\ldots$$
$$=\frac{\pi^2}{16}-2\left(\frac{\cos 2x}{2^2}+\frac{\cos 6x}{6^2}+\frac{\cos 10x}{10^2}+\ldots\right).$$

Graph. As discussed before, here the graph will be symmetrical about y-axis because if we change x into $-x$, y does not change. $\cos(-\theta)=\cos\theta$.

Ex. 29. *Find a series of (a) sines of multiples of x, (b) cosine of multiples of x which will represent x in the interval $0 \leqslant x \leqslant \pi$. Show by a graph the nature of the series for all values of x.* **(Rajputana 56)**

Sine Series $f(x)=\sum_{1}^{\infty} b_n \sin nx$.

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=-\frac{2}{\pi}\cdot\frac{\pi}{n}(-1)^n=\frac{2}{n}(-1)^{n+1}$$

$$\therefore\quad b_1=\tfrac{2}{1},\ b_2=-\tfrac{2}{2},\ b_3=\tfrac{2}{3}\ldots$$

$\therefore\quad f(x)=2\,[\sin x-\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x-\tfrac{1}{4}\sin 4x\ldots]$

In the interval 0 to π, the graph is given by $y=x$.

Again if $\pi < x \leqslant 2\pi$, suppose $x=2\pi-x'$, so that $0 < x' < \pi$.

$\therefore$ Series $=2\,[\sin(2\pi-x')-\tfrac{1}{2}\sin(4\pi-2x')+\ldots]$

$\quad=-2\,[\sin x'-\tfrac{1}{2}\sin 2x'+\tfrac{1}{3}\sin 3x'\ldots]$

$\quad=-x'=x-2\pi,$

so that y is $-$ive when $\pi < x < 2\pi$.

Again if if $2\pi < x < 3\pi$, suppose $x=2\pi+x''$, so that $0 < x'' < \pi$.

$\therefore$ Series $=2\,[\sin(2\pi+x'')-\tfrac{1}{2}\sin(4\pi+2x'')\ldots]$

$\quad=2\,[\sin x''-\tfrac{1}{2}\sin 2x''\ldots]$

$\quad=x''=x-2\pi,$

so that y is $+$ive when $2\pi < x < 3\pi$.

Also if x be changed into $-x$, y changes to $-y$, so that there is symmetry in opposite quadrants ; hence the graph of the curve is as shown below :—



Cosine Series. $f(x)=a_0+\sum_{1}^{\infty} a_n \cos nx$,

$$a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{\pi} x\,dx=\frac{1}{\pi}\cdot\frac{\pi^2}{2}=\frac{\pi}{2}.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[x\,\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}=\frac{2}{\pi}\cdot\frac{1}{n^2}\,[(-1)^n-1].$$

If n be even, then $a_n=0$; $\therefore\ a_2=a_4=\ldots=0.$

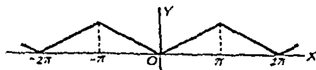
If n be odd, then $a_n=\frac{2}{\pi}\cdot\frac{1}{n^2}\cdot(-2)=-\frac{4}{\pi}\cdot\frac{1}{n^2}$

$$\therefore\quad f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right].$$

At $x=0$, the sum of the cosine series is $f(+0)=0$.

$$\therefore\quad 0=\frac{\pi}{2}-\frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]\ \text{ or }\ \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

Here the symmetry will be about y-axis. Hence the graph of the series is as shown below :



Ex. 30. *If* $-\frac{\pi}{2}<x<\frac{\pi}{2}$, *show that*

$$x=\frac{4}{\pi}\left(\sin x-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}\ldots\right).$$

Let $z=\frac{\pi}{2}+x$, so that when x varies from $-\frac{\pi}{2}$ to π, z varies from 0 to π.

$\therefore$ Let $z=a_0+\sum_{1}^{\infty} a_n \cos nz$ in the interval $(0, \pi)$.

$$\therefore \quad = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos z + \frac{1}{3^2}\cos 3z + \frac{1}{5^2}\cos 5z + \ldots\right]$$

[by Q. 29].

Now replace z by $\frac{\pi}{2} + x$.

$\therefore \quad \cos z = -\sin x, \cos 3z = \sin 3x, \cos 5z = -\sin 5z.$

$$\therefore \quad \frac{\pi}{2} + x = \frac{\pi}{2} - \frac{4}{\pi}\left[-\sin x + \frac{1}{3^2}\sin 3x - \frac{1}{5^2}\sin 5x + \ldots\right]$$

or
$$x = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right].$$

Ex. 31. *If the function defined by $y = x^2$ from 0 to $\frac{\pi}{2}$ and by $y = 0$ from $\frac{\pi}{2}$ to π be represented by the series of sines of multiplies of x, show that coefficient of $\sin nx$ is*

$$\left(\frac{4}{\pi n^3} - \frac{\pi}{2n}\right)\cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2} - \frac{4}{\pi n^3}.$$

To what value does the series converge at the point $x = \frac{\pi}{2}$?

Sketch the graph of the function represented by the series for values of x not restricted to lie between 0 and π, and also indicate the graph of the cosine series which represents the same function in the interval 0 to π. **(Agra 60)**

$$f(x) = x^2,\ 0 < x < \frac{\pi}{2};\ f(x) = 0,\ \frac{\pi}{2} < x < \pi.$$

Let $f(x) = \sum_{1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

or $b_n = \frac{2}{n}\int_0^{\pi/2} x^2 \sin nx\, dx + 0$ by the definition of the function

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2}{4n}\cos\frac{n\pi}{2}+2\,\frac{\pi}{2n^2}\sin\frac{n\pi}{2}+\frac{2}{n^3}\left(\cos\frac{n\pi}{2}-1\right)\right].$$

$$b_n=\left(\frac{4}{\pi n^3}-\frac{\pi}{2n}\right)\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}-\frac{4}{\pi n^3},$$

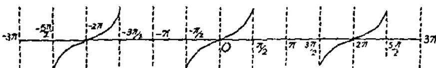
where b_n is the coefficient of $\sin nx$.

Again we know from § 5 P. 244 that the series converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and π.

$\because$ Sum at $x=\frac{\pi}{2}$ is $\frac{1}{2}\left[0+\left(\frac{\pi}{2}\right)^2\right]=\frac{\pi^2}{8}$.

$\because$ $f\left(\frac{\pi}{2}+0\right)=0$ by definition.

In the interval $(0, \pi)$ the function represents a parabola $y=x^2$, $0 \leqslant x \leqslant \frac{\pi}{2}$ and the line $y=0$, *i.e.* axis of x, $\frac{\pi}{2} < x < \pi$.



Note. Treat the part of x-axis between $\frac{\pi}{2}$ to π and $\frac{5\pi}{2}$ to 3π as black and similarly on the $-$ive side forming part of the graph.

Again if $\pi < x < 3\pi/2$, let $x=2\pi-x'$,

so that $\qquad \pi/2 < x' < \pi$.

Then the series is zero (given) so that $y=0$.

If $\frac{3\pi}{2} < x < 2\pi$, let $x=2\pi-x'$, so that $0 < x' < \frac{\pi}{2}$.

Replace x by $2\pi x'$, *i.e.* $\sin x$ by $-\sin x'$, $\sin 2x$ by $-\sin 2x'$ and so on, so that the series becomes

$-x'^2=-(2\pi-x)^2$, *i.e.* $y=-(2\pi-x)^2$ (parabolic),

$\because \quad x^2=\Sigma b_n \sin nx$.

Again if $2\pi < x < \frac{5\pi}{2}$, let $x=2\pi+x'$, so that $0 < x < \frac{\pi}{2}$.

Replacing x by $2\pi+x''$, *i.e.* $\sin x$ by $\sin(2\pi+x'')$, *i.e.* $\sin x''$ and so on, the series becomes $x''^2=(x-2\pi)^2$, *i.e.* $y=(x-2\pi)^2$ (parabolic)

Again if $\frac{5\pi}{2}<x<3\pi$, let $x=2\pi+x''$, so that $\frac{\pi}{2}<x''<\pi$

and by definition, series$=0$. *i.e.* $y=0$

Again we know that in the case of sine series there is symmetry in opposite quadrants. Hence the graph is as shown,

Graph of cosine series.

In the interval 0 to π, the graph is given by $y=x^2$, $0<x<\pi/2$ and $y=0$, $\pi/2<x<\pi$.

Again if $\pi<x<\frac{3\pi}{2}$ let $x=2\pi-x'$, so that $\frac{\pi}{2}<x<\pi$.

then $y=0$

If $\frac{3\pi}{2}<x<2\pi$, let $x=2\pi-x'$, so that $0<x'<\frac{\pi}{2}$; then $\cos x=\cos(2\pi-x')=\cos x'$, $\cos 2x=\cos(4\pi-2x')=\cos 2x'$, so that the cosine series does not change, *i.e.* $y=x'^2=(2\pi-x)^2$ (parabolic).

If $2\pi<x<\frac{5\pi}{2}$, let $x=2\pi+x''$, so that $0<x''<\frac{\pi}{2}$.

Here also cosine series does not change if x be replaced by $2\pi+x''$.

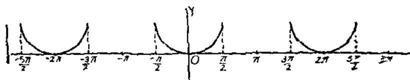
$$\therefore\quad y=x''^2=(x-2\pi)^2 \text{ (parabolic).}$$

If $\frac{5\pi}{2}<x<3\pi$, let $x=2\pi+x''$, so that $\frac{\pi}{2}<x''<\pi$.

Cosine series does not change and $y=0$ by definition and so on.

Also we know that in cosine series the symmetry is about axis of y Hence the graph of cosine series is as

shown below .



Note. Treat the part of x-axis between $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$ to 3π as black and similarly on $-$ive side forming part of the graph.

Ex. 32. *Expand e^{ax} in a series of multiples of x, $0 < x < \pi$, and examine the series obtained.*

$$e^{ax}=\sum_{1}^{\infty} b_n \sin nx,$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi} e^{ax} \sin nx \, dx=\frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a \sin nx-n \cos nx)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\cdot\frac{n}{a^2+n^2}\{1-(-1)^n e^{a\pi}\}. \quad \because \cos n\pi=(-1)^n.$$

$$\therefore \quad b_1=\frac{2}{\pi}\cdot\frac{1-e^{a\pi}}{a^2+1^2},\ b_2=\frac{2}{\pi}\cdot\frac{2(1-e^{a\pi})}{a^2+2^2},\ldots$$

$$\therefore \quad e^{ax}=\frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1}\sin x+\frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x\right.$$

$$\left.+\frac{3(1-e^{a\pi})}{a^2+3^2}\sin 3x+\ldots\right]. \quad \ldots(1)$$

Also from § 7 P. 244, we know that sum of the series is zero at $x=0$ or π.

In the interval $(0, \pi)$ the series is given by

$$y=e^{ax} \text{ and at } x=\pi, y=0.$$

If $\pi < x < 2\pi$, let $x=2\pi-x'$, so that $0 < x' < \pi$ ... $\sin x=\sin(2\pi-x')=-\sin x'$ and $\sin 2x=-\sin 2x'$ and so on so that $y=-e^{ax'}$ from (1) or $y=-e^{a(2\pi-x)}$

Similarly if $2\pi < x < 3\pi$, then put $x=2\pi+x''$ so that $0 < x'' < \pi$.

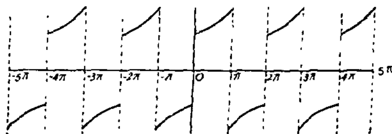
Also $\sin x=\sin x''$, $\sin 2x=\sin 2x''$, so that

$$y=e^{ax''}=e^{a(x-2\pi)}.$$

We have shown that at $x=0, \pi, 2\pi\ldots$, sum of the series is zero.

Also we know that sine series is symmetrical in opposite quadrants

Therefore the graph of the series is as under :



Ex. 33. *Given that $F(x)$ is an odd function of x, equal to x in the interval $\left(0, \frac{\pi}{2}\right)$ and equal to $\left(\frac{\pi}{2}-x\right)$ in the interval $\left(\frac{\pi}{2}, \pi\right)$; expand $F(x)$ in a Fourier's series valid for the interval $(-\pi, \pi)$.*

Draw a graph of the sum y of the series for the interval $(-3\pi, 3\pi)$. **(Rajputana 51)**

$$F(x)=x, \qquad 0 < x < \frac{\pi}{2}$$

$$=\frac{\pi}{2}-x, \quad \frac{\pi}{2} < x < \pi.$$

Since $F(x)$ is an odd function of x and hence by § 4 P. 235 the series will be a sine series.

$$\therefore \quad F(x)=\sum_{1}^{\infty} b_n \sin nx,$$

where $b_n=\dfrac{2}{\pi}\displaystyle\int_0^{\pi} F(x) \sin nx\, dx$

$$=\frac{2}{\pi}\left[\int_0^{\frac{1}{2}\pi} x \sin nx\, dx+\int_{\frac{1}{2}\pi}^{\pi}\left(\frac{\pi}{2}-x\right) \sin nx\, dx\right.$$

$$=\frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\frac{1}{2}\pi}\right.$$

$$\left.+\left\{\left(\frac{\pi}{2}-x\right)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\frac{1}{2}\pi}^{\pi}\right]$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{1}{n^2}\sin\frac{n\pi}{2}+\frac{\pi}{2n}\cos n\pi+\frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{2}{\pi}\cdot\left[\frac{2}{n^2}\sin\frac{n\pi}{2}-\frac{\pi}{2n}\left(\cos\frac{n\pi}{2}-\cos n\pi\right)\right]$$

$$=\frac{2}{\pi}\cdot\frac{1}{n^2}\left[2.\sin 2\,\frac{n\pi}{4}\cos\frac{n\pi}{4}-\frac{\pi n}{2}.2\sin\frac{3n\pi}{4}\sin\frac{n\pi}{4}\right]$$

$$=\frac{2}{\pi}\,\frac{1}{n^2}\left[4\cos\frac{n\pi}{4}-n\pi\sin\frac{3n\pi}{4}\right]\sin\frac{n\pi}{4}.$$

$$\therefore \quad F(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[4\cos\frac{n\pi}{4}-n\pi\sin\frac{3n\pi}{4}\right]\sin\frac{n\pi}{4}\sin nx.$$

In the interval $(0, \pi)$, the graph represents the lines

$$y=x, \qquad 0 \leqslant x < \frac{\pi}{2}$$

and $$y=\frac{\pi}{2}-x, \quad \frac{\pi}{2} < x < \pi.$$

Now we shall consider other values of x.

If $\pi < x < \dfrac{3\pi}{2}$, put $x=2\pi-x'$, so that $\dfrac{\pi}{2} < x' < \pi$.

Then $F(x)$ becomes $-F(x')$.

$$\therefore \quad \sin x=\sin(2\pi-x')=-\sin x'$$

or $$y=-\left[\frac{\pi}{2}-x'\right]=-\frac{\pi}{2}+x'=-\frac{\pi}{2}+2\pi-x=\frac{3\pi}{2}-x.$$

If $\frac{3\pi}{2}<x<2\pi$, put $x=2\pi-x'$, so that $0<x'<\frac{\pi}{2}$ and $y=-x'$ as $F(x)$ becomes $-F(x')$ or $y=x-2\pi$.

If $2\pi<x<5\frac{\pi}{2}$, let $x=2\pi+x''$, so that $0<x''<\frac{\pi}{2}$.

Also $$\sin x=\sin(2\pi+x'')=\sin x''.$$

$\therefore$ $F(x)$ remains unchanged.

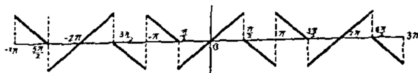
$$\therefore\quad y=x''=x-2\pi,$$

If $\frac{5\pi}{2}<x<3\pi$ let $x=2\pi+x''$, so that $\frac{\pi}{2}<x''<\pi$.

$$\therefore\quad y=\frac{\pi}{2}-x''=\frac{\pi}{2}-(x-2\pi)=\frac{5\pi}{2}-x$$

Similarly we may consider the interval on the other side ; *but in sine series we know that there is always symmetry* in opposite quadrants. Hence the graph is as shown below. Clearly the points of discontinuity are

$$x=\pm\frac{\pi}{2},\ \pm\pi,\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \pm3\pi.$$



Ex. 34. *Show that*

$$\frac{\pi}{4}\cos x=\frac{2}{1.3}\sin 2x+\frac{4}{3.5}\sin 4x+\frac{6}{5.7}\sin 6x+\dots$$
$$\dots+\frac{2n}{(2n-1)(2n+1)}\sin 2nx+\dots$$

and examine what is the sum of the series for other values of x. Show by a graph the nature of the series.

$$\frac{\pi}{4}\cos x=\sum_{1}^{\infty} b_n \sin nx, \text{ where } b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \; dx$$

or $$b_n=\frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\cos x.\sin nx \; dx$$

$$=\frac{1}{4}\int_0^{\pi}[\sin (n+1)x+\sin (n-1)x]\,dx$$

$$=\frac{1}{4}\left[-\frac{\cos (n-1)x}{n-1}-\frac{\cos (n+1)x}{n+1}\right]_0^{\pi}$$

$$=\frac{1}{4}\left[-\frac{\cos (n-1)\pi-1}{n-1}-\frac{\cos (n+1)\pi-1}{n+1}\right]$$

$$=\frac{1}{4}\left[\left(\frac{1}{n+1}+\frac{1}{n-1}\right)+\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\cos n\pi\right]$$

$$\because \quad \cos (n\pm 1)\pi=\cos n\pi\cos\pi+\sin n\pi\sin\pi$$

$$=-\cos n\pi$$

$$=\frac{1}{4}\cdot\frac{2n}{n^2-1}(1+\cos n\pi)=\frac{n}{2(n^2-1)}[1+(-1)^n].$$

If n be odd, then $(-1)^n=-1$.

$$\therefore \quad b_n=0, \quad i.e. \quad b_1=b_3=b_5=\ldots=0.$$

If n be even, then $(-1)^n=1$.

$$\therefore \quad b_n=\frac{n}{2(n^2-1)}.2=\frac{n}{(n-1)(n+1)}.$$

$$\therefore \quad b_2=\frac{2}{1.3}, b_4=\frac{4}{3.5}, b_6=\frac{6}{5.6} \text{ and so on.}$$

$$\therefore \quad \frac{\pi}{4}\cos x=\frac{2}{1.3}\sin 2x+\frac{4}{3.5}\sin 4x+\frac{6}{5.7}\sin 6x+\ldots$$

The general term, when n=even=$2k$, say, is

$$\frac{2k}{(2k-1)(2k+1)}\sin 2kx,$$

If $\pi < x < 2\pi$, put $x=2\pi-x'$; $\therefore \; 0 < x' < \pi$ and $\sin x=\sin (2\pi-x')=-\sin x'$, $\sin 2x=-\sin 2x'$ and so on.

$\therefore \; f(x)$ becomes $-f(x')=-\frac{\pi}{4}\cos x'=-\frac{\pi}{4}\cos (2\pi-x)$

$$=-\frac{\pi}{4}\cos x.$$

$$\therefore \quad y=-\frac{\pi}{4}\cos x$$

If $2\pi < x < 3\pi$, put $x=2\pi+x''$, so that $0 < x'' < \pi$. Here $\sin x=\sin x''$ and so on.

$\therefore \quad f(x)$ becomes $f(x'')=\frac{\pi}{4}\cos x''=\frac{\pi}{4}\cos(x-2\pi)$.

$$\therefore \quad y=\frac{\pi}{4}\cos x.$$

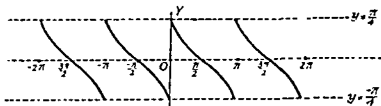
Similarly, if $3\pi < x < 4\pi$, then putting $x=4\pi-x''$, so that $0 < x'' < \pi$, $\sin x=-\sin x''$ and so on.

$$y=-\frac{\pi}{4}\cos x''=-\frac{\pi}{4}\cos(4\pi-x)=-\frac{\pi}{4}\cos x.$$

Also we know that in the case of sine series there is symmetry in opposite quadrants.

$y=\frac{\pi}{4}$ when $x=0$; $y=0$, when $x=\frac{\pi}{2}$; $y=-\frac{\pi}{4}$ when $x=\pi$,

and $y=\frac{\pi}{4}\cos x$ when $0 < x < \pi$.



Ex. 35. *Show that if $0 < x < \pi$, then*

$$\frac{\pi}{4}\sin x=\frac{1}{2}-\frac{\cos 2x}{1.3}-\frac{\cos 4x}{3.5}-\frac{\cos 6x}{5.7}-\ldots$$

Show by graph the nature of series for all values of x.

$\frac{\pi}{4}\sin x$ when expanded in cosines of multiples of x.

$$=a_0+\sum_1^\infty a_n \cos nx$$

where $a_0=\frac{1}{\pi}\int_0^\pi f(x)=\frac{1}{\pi}\int_0^\pi \frac{\pi}{4}\sin x\,dx=-\frac{1}{4}\left[\cos x\right]_0^\pi$

$$=-\tfrac{1}{4}(-2)=\tfrac{1}{2}.$$

$$a_n=\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^\pi \frac{\pi}{4}\sin x.\cos nx\,dx$$

$$=\frac{1}{4}\int_0^\pi [\sin(n+1)x-\sin(n-1)x]\,dx$$

$$=\frac{1}{4}\left[\frac{\cos(n-1)x}{n-1}-\frac{\cos(n+1)x}{n+1}\right]_0^\pi$$

$$=\frac{1}{4}\left[\frac{\cos(n-1)\pi-1}{n-1}-\frac{\cos(n+1)\pi-1}{n+1}\right]$$

$$=\frac{1}{4}\left[\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\cos n\pi\right],$$

$\because$ $\cos(n\pm1)\pi=-\cos n\pi$ as in last question

$$=\frac{1}{4}\cdot\frac{-2}{n^2-1}[1+\cos n\pi]=-\frac{1}{2(n^2-1)}\{1+(-1)^n\}$$

If n be odd, then $(-1)^n=-1$ *i.e.* $a_n=0$.

$$\therefore\quad a_1=a_3=a_5=\ldots=0$$

If n be even, then $(-1)^n=1$,

i.e. $$a_n=-\frac{1}{(n^2-1)}=-\frac{1}{(n-1)(n+1)}.$$

$$\therefore\quad a_2=-\frac{1}{1.3},\ a_4=-\frac{1}{3.5},\ a_6=-\frac{1}{5.7}.$$

$$\therefore\ \frac{\pi}{4}\sin x=\frac{1}{2}-\frac{1}{1.3}\cos 2x-\frac{1}{3.5}\cos 4x-\frac{1}{5.7}\cos 6x-\ldots$$

Again $\pi<x<2\pi$; let $x=2\pi-x'$ so that $0<x'<\pi$; then $\cos 2x=\cos(4\pi-2x')=\cos 2x'$ so that $f(x)$ becomes $f(x')$ *i.e.* $y=\frac{\pi}{4}\sin x'=\frac{\pi}{4}\sin(2\pi-x)=-\frac{\pi}{4}\sin x$.

If $2\pi < x < 3\pi$, let $x = 2\pi + x''$ so that $0 < x'' < \pi$.

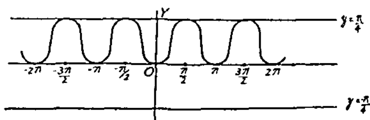
Also $\cos 2x = \cos(4\pi + 2x'') = \cos 2x''$ so that $f(x)$ becomes $f(x'')$ *i.e.* $y = \frac{\pi}{4}\sin x'' = \frac{\pi}{4}\sin(x - 2\pi) = \frac{\pi}{4}\sin x$

If $3\pi < x < 4\pi$, let $x = 4\pi - x'''$ so that $0 < x''' < \pi$.

Also $\cos 2x = \cos(8\pi - 2x''') = \cos 2x'''$ *i.e.* $f(x)$ becomes $f(x''')$ *i.e.* $y = \frac{\pi}{4}\sin x''' = \frac{\pi}{4}\sin(4\pi - x) = -\frac{\pi}{4}\sin x$ and so on

Again we know that the graph of cosine series is symmetrical about y-axis.

Hence the graph of the series is as shown below :



§ 6. Fourier's Series in the interval $(0, 2\pi)$.

To prove that

$$f(x) = a_0 + \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

where $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx,\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx.$$

In order to find a_0, we integrate both sides of (1) within limits 0 to 2π.

$$\therefore \int_0^{2\pi} f(x)\,dx = a_0 \int_0^{2\pi} dx + \sum_{n=1}^{\infty} \left\{ a_n \int_0^{2\pi} \cos nx\,dx \right\} + \sum_{n=1}^{\infty} \left\{ b_n \int_0^{2\pi} \sin nx\,dx \right\}$$

or

$$\int_0^{2\pi} f(x)\,dx = a_0\, 2\pi + 0 + 0 \text{ by § 1·4}$$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^{2\pi} f(v)\,dv. \qquad \ldots(2)$$

Again in order to find a_n, we multiply both sides of (1) by $\cos nx$ and integrate between the limits 0 to 2π

$$\therefore \int_0^{2\pi} f(x) \cos nx\,dx$$

$$= a_0 \int_0^{2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} \left[a_n \int_0^{2\pi} \cos^2 nx\,dx \right] + \sum_{n=1}^{\infty} \left[\frac{1}{2} b_n \int_0^{2\pi} \sin 2nx\,dx \right]$$

$$= 0 + a_n . \pi + 0 \text{ by § 1·2 and 1·5.}$$

$$\therefore \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{2\pi} f(v) \cos nv\,dv. \qquad \ldots(3)$$

Similarly to find b_n, we multiply both sides of (1) by $\sin nx$ and integrate within limits 0 to 2π.

$$\therefore \int_0^{2\pi} f(x) \sin nx\,dx$$

$$= a_0 \int_0^{2\pi} \sin nx\,dx + \sum_{n=1}^{\infty} \left[a_n \int_0^{2\pi} \frac{1}{2} \sin 2nx\,dx \right] + \sum_{1}^{\infty} \left[b_n \int_0^{2\pi} \sin^2 nx\,dx \right]$$

$$= 0 + 0 + b_n . \pi.$$

$$\therefore \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \int_0^{2\pi} f(v) \sin nv\,dv. \qquad \ldots(4)$$

Hence $f(x)=a_0+\sum_{1}^{\infty}(a_n \cos nx+b_n \sin nx)$,

where a_0, a_n, b_n have values written above in (2), (3) and (4).

The sum of the above series is $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and 2π and is $\frac{1}{2}[f(2\pi-0)+f(+0)]$ at $x=0$ and $x=2\pi$.

Particular Case.

We know that
$$\left.\begin{array}{l}\cos(2n\pi-\theta)=\cos\theta \\ \text{and} \quad \sin(2n\pi-\theta)=-\sin\theta\end{array}\right\} \qquad \ldots(1)$$

Case I. $f(2\pi-x)=f(x)$. $\qquad$ ...(2)

Then, $b_n=\frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx$

$$=\frac{1}{\pi}\int_0^{2\pi} f(2\pi-x) \sin(2n\pi-nx), \quad \text{by Prop. IV}$$

i.e., $$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

or $\quad b_n=-\frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx$ by (1) and (2) $=-b_n$.

$$\therefore \quad 2b_n=0 \quad \text{or} \quad b_n=0.$$

Hence in this case the series will reduce to a cosine series,

i.e., $$f(x)=a_0+\sum_{1}^{\infty} a_n \cos nx.$$

Case II. $f(2\pi-x)=-f(x)$. $\qquad$ ...(3)

Then, $a_0=\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{2\pi}\int_0^{2\pi} f(2\pi-x)\,dx$

or $$a_0=-\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \quad \text{by (3)}=-a_0.$$

$$\therefore \quad 2a_0=0 \quad \text{or} \quad a_0=0.$$

Similarly as in case I by the help of relation (1), we can show that $a_n=0$ and hence in this case the series will reduce to purely sine series,

i.e. $$f(x)=\sum_{1}^{\infty} b_n \sin nx$$

In general, if $\alpha < x < 2\pi+\alpha$,

i.e. length of interval is 2π, then

$$f(x)=a_0+\sum_{1}^{\infty} (a_n \cos nx+b_n \sin nx),$$

where $$a_0=\frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx,$$

$$a_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\,dx$$

and $$b_n=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\,dx.$$

Ex. 36. *Prove that the series*

$$2\left[\sin x+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right]$$

represents $(\pi-x)$ *in the interval* $0 < x < 2\pi$.

Here $$f(x)=(\pi-x)$$

and $$f(2\pi-x)=\pi-(2\pi-x)=-\pi+x=-(\pi-x),$$

i.e. $$f(2\pi-x)=-f(x)$$

and hence the series will be a sine series.

$$\therefore\ f(x)=\sum_{1}^{\infty} b_n \sin nx\,dx,$$

where $$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi} (\pi-x) \sin nx\,dx$$

$$=\frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{1}{n}\{-\pi\,(-\cos 2n\pi)-\pi\,(-1)\}\right]$$

$$=\frac{1}{n}\,[\cos 2n\pi+1]=\frac{2}{n},\quad \because\quad \cos 2n\pi=1.$$

$$\therefore\quad b_1=\tfrac{2}{1},\ b_2=\tfrac{2}{2},\ b_3=\tfrac{2}{3}\ \ .$$

$$\therefore\quad f(x)=(\pi-x)=2\,[\sin x+\tfrac{1}{2}\sin 2x+\tfrac{1}{3}\sin 3x+\ \ldots].$$

Also the sum of series when $x=0$ or 2π is

$$\tfrac{1}{2}\,[f(2\pi-0)+f(+0)]$$
$$=\tfrac{1}{2}\,[(\pi-2\pi)+(\pi-0)]=0.$$

Ex. 37. *Prove that if $0<x<2\pi$,*

$$\frac{\pi}{2a}\,\frac{\cosh a\,(\pi-x)}{\sinh a\pi}=\frac{1}{2a^2}+\frac{\cos x}{a^2+1^2}+\frac{\cos 2x}{a^2+2^2}+\frac{\cos 3x}{a^2+3^2}+\ \ldots$$

Now

$$\cosh a\,\{\pi-(2\pi-x)\}=\cosh a\,(-\pi+x)=\cosh a\,(\pi-x)\},$$

$$\because\quad \cosh(-\theta)=\cosh\theta,$$

$$\therefore\quad f(2a-x)=f(x)$$

and hence in this case the series will be a cosine series.

$$f(x)=a_0+\sum_{1}^{\infty}a_n\cos nx.$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{2\pi}\int_0^{2\pi}\frac{\pi}{2a}\,\frac{\cosh a\,(\pi-x)}{\sinh a\pi}\,dx$$

$$=\frac{1}{4a\sinh a\pi}\left[\frac{\sinh a\,(\pi-x)}{-a}\right]_0^{2\pi}$$

$$=-\frac{1}{4a^2\sinh a\pi}\,[\sinh(-a\pi)-\sinh a\pi].$$

But $\qquad\sinh(-\theta)=-\sinh\theta.$

$$\therefore\quad a_0=-\frac{1}{4a^2\sinh a\pi}\cdot(-2\sinh a\pi)=\frac{1}{2a^2},$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\cdot\frac{\pi}{2a \sinh a\pi}\int_0^{2\pi}\left\{\frac{e^{a(\pi-x)}+e^{-a(\pi-x)}}{2}\right\}\cos nx\, dx$$

$$=\frac{1}{4a \sinh a\pi}\int_0^{2\pi}(e^{a\pi}.e^{-ax}\cos nx+e^{-a\pi}.e^{ax}\cos nx)\, dx$$

$$=\frac{1}{4a \sinh a\pi}\cdot\frac{1}{a^2+n^2}\Big[e^{a\pi}.e^{-ax}(-a\cos nx-n\sin nx)$$

$$+e^{-a\pi}.e^{ax}(a\cos nx+n\sin nx)\Big]_0^{2\pi}$$

Now $\sin 2n\pi=0$, $\cos 2n\pi=1$.

$$\therefore\quad a_n=\frac{1}{4a\sinh a\pi.(a^2+n^2)}\left[\{e^{a\pi}.e^{-2\pi a}.(-a)+e^{-a\pi}.e^{2\pi a}(a)\}\right.$$

$$\left.-\{e^{a\pi}.1.(-a)+e^{-a\pi}\ 1.(a)\}\right]$$

$$=\frac{1}{4a\sinh a\pi.(a^2+n^2)}[2a\,(e^{a\pi}-e^{-a\pi})]$$

$$=\frac{1}{4a\sinh a\pi.(a^2+n^2)}[2a\ 2\sinh a\pi]=\frac{1}{a^2+n^2}.$$

$$\therefore\quad a_1=\frac{1}{a^2+1^2},\ a_2=\frac{1}{a^2+2^2},\ a_3=\frac{1}{a^2+3^2}+\ ..$$

$$\therefore\quad f(x)=\frac{1}{2a^2}+\frac{\cos x}{a^2+1^2}+\frac{\cos 2x}{a^2+2^2}+\frac{\cos 3x}{a^2+3^2}+\dots$$

Ex. 38. *Show that if* $0 < x < 2\pi$,

$$\frac{\pi}{2}\cdot\frac{\sinh a(\pi-x)}{\sinh a\pi}=\frac{\sin x}{a^2+1^2}+\frac{2\sin 2x}{a^2+2^2}+\dots$$

$$\sinh a\,[\pi-(2\pi-x)]=\sinh a\,(-\pi+x)\ \sinh\{-a\,(\pi-x)\}$$

$$=-\sinh a\,(\pi-x),$$

i.e. $$f(x)=-f(x)$$

and hence in this case the series will be a sine series.

$$\therefore\quad f(x)=\sum^{\infty} b_n \sin nx,$$

where

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\cdot\frac{\pi}{2\sinh a\pi}\int_0^{2\pi}\frac{e^{a(\pi-x)}-e^{-a(\pi-x)}}{2}\sin nx\,dx$$

$$=\frac{1}{4\sinh a\pi}\int_0^{2\pi} e^{a\pi}.e^{-ax}\sin nx-e^{-a\pi}.e^{ax}\sin nx\,dx$$

$$=\frac{1}{4\sinh a\pi}\cdot\frac{1}{a^2+n^2}\Big[\{e^{a\pi}.e^{-ax}(-a\sin nx-n\cos nx)\}$$

$$-e^{-a\pi}e^{ax}(a\sin nx-n\cos nx)\Big]_0^{2\pi}$$

$$=\frac{1}{4\sinh a\pi.(a^2+n^2)}[\{e^{a\pi}.e^{-2\pi a}(-n)-e^{-a\pi}.e^{2\pi a}(-n)\}]$$

$$-\{e^{a\pi}.1.(-n)-e^{-a\pi}.1\,(-n)\}]$$

$$=\frac{n}{4\sinh a\pi.(a^2+n^2)}[-e^{-\pi a}+e^{\pi a}+e^{a\pi}-e^{-\pi a}]$$

$$=\frac{n}{4\sinh a\pi.(a^2+n^2)}[2(e^{\pi a}-e^{-\pi a})]$$

$$=\frac{n}{4\sinh a\pi.(a^2+n^2)}.(2.2\sinh a\pi)=\frac{n}{a^2+n^2}.$$

$\therefore\quad b_1=\frac{1}{a^2+1^2}, b_2=\frac{2}{a^2+2^2}, b_3=\frac{3}{a^2+3^2};\quad\therefore\quad f(x)=$ as given.

§ 6. Miscellaneous forms.

Fourier's Series for interval (−1, 1).

We are to find a Fourier's series for a given function of x defined in the interval $(-l, -l)$ subject to conditions laid in § 2 P. 220.

Let us suppose $y=\frac{\pi x}{l}$, so that as x varies from $-l$ to l, y varies from $-\pi$ to π. Also $f(x)=f\left(\frac{ly}{\pi}\right)=F(y)$ say.

Now we know that in the interval $(-\pi, \pi)$,

$$F(y)=a_0+\sum_1^{\infty}(a_n \cos ny+b_n \sin ny),$$

where $$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} F(y)\, dy=\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) dy.$$

Put $y=\frac{\pi x}{l}$, *i.e.*, $dy=\frac{\pi}{l}\, dx$ and adjust the limits.

$$\therefore \quad a_0=\frac{1}{2\pi}\int_{-l}^{l} f(x)\,\frac{\pi}{l}\, dx=\frac{1}{2l}\int_{-l}^{l} f(x)\, dx$$

Similarly, $$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

and $$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx.$$

Hence $$f(x)=a_0+\sum_1^{\infty}\left(a_n \cos\frac{n\pi x}{l}+b_n \sin\frac{n\pi x}{l}\right).$$

Note. *If you compare the values of a_0, a_n, b_n with those for interval $(-\pi, \pi)$, you will observe a similarity, i.e., instead of π, we have l and angle nx with every cos or sin is to be replaced by $\frac{n\pi x}{l}$.*

Similarly the above series is $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between $-l$ to l and is $\frac{1}{2}[f(l-0)+f(-l+0)]$ for $x=-l$ and $x=l$ and it is periodic whose period is $2l$.

Particular case. Range $(0, l)$.

Sine Series. $f(x)=\sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$,

where $$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\, dx.$$

Cosine Series. $f(x)=a_0+\sum_1^{\infty} a_n \cos\frac{n\pi x}{l}$,

where $$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\, dx.$$

and $$a_0=\frac{1}{l}\int_0^l f(x)\,dx.$$

Compare the above results with the corresponding results for range $(0, \pi)$

Ex. 39. *Show that the series* $\frac{4}{\pi}\left(\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\ldots\right)$ *is equal to 1 when* $0 < x < l$.

Here the interval is $(0, l)$ and $f(x)=1$.

Sine series is $f(x)=1=\Sigma b_n \sin\frac{n\pi x}{l}$,

where
$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$
$$=\frac{2}{l}\int_0^l 1\,\sin\frac{n\pi x}{l}\,dx=\frac{2}{l}\left(\frac{-l}{n\pi}\right)\left[\cos\frac{n\pi x}{l}\right]_0^l$$
$$=-\frac{2}{n\pi}[\cos n\pi-1]=\frac{2}{n\pi}[1-(-1)^n].$$

When n is even, then $(-1)^n=1$;

$\therefore\ b_n=0$, *i.e.*, $b_2=b_4=\ldots=0$

and when n is odd, then $(-1)^n=-1$. $\therefore\ b_n=\frac{4}{n\pi}$.

$$\therefore\ b_1=\frac{4}{\pi},\ b_3=\frac{4}{3\pi},\ b_5=\frac{4}{5\pi}.$$

$$\therefore\ f(x)=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}\ldots\right].$$

Ex. 40. *If* $f(x)=kx$ *when* $0 < x < l/2$,
$=k(l-x)$ *when* $l/2 < x < l$,

prove that
$$f(x)=\frac{kl}{4}-\frac{8kl}{\pi^2}\left[\frac{1}{2^2}\frac{\cos 2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\frac{1}{10^2}\cos\frac{10\pi x}{l}+\ldots\right].$$

In the interval $(0, l)$,
$$f(x)=a_0+\sum_1^\infty a_n\cos\frac{n\pi x}{l}.$$

where $a_n=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}\left[\int_0^{l/2} kx\,dx+\int_{l/2}^{l} k\,(l-x)\,dx\right]$

$$=\frac{1}{l}\left[k\cdot\frac{l^2}{8}+kl\cdot\frac{l}{2}-k\cdot\frac{l^2}{2}\,(1-\tfrac{1}{4})\right]=\frac{1}{l}\cdot\frac{kl^2}{8}\,[1+4-3]=\tfrac{1}{4}kl.$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\left[\int_0^{l/2} kx\,\frac{\cos n\pi x}{l}\,dx+\int_{l/2}^{l} k\,(l-x)\,\frac{\cos n\pi x}{l}\,dx\right]$$

$$=\frac{2k}{l}\left[\left\{x\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(1)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_0^{l/2}+(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\Bigg\}_{l/2}^{l}\right]$$

$$=\frac{2k}{l}\left[\left\{\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right\}+\left\{0-\frac{l^2}{2n\pi}\sin\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right\}\right]$$

$$=\frac{2k}{l}\cdot\frac{l^2}{n^2\pi^2}\left[2\cos\frac{n\pi}{2}-\cos n\pi-1\right].$$

n odd. $\cos\frac{n\pi}{2}=0$, $\cos r\pi=-1$. $\therefore\ a_n=0.$

$$\therefore\ a_1=a_3=a_5=\ldots=0.$$

n even but multiple of four $=4r$ say.

$$\cos\frac{n\pi}{2}=\cos 2r\pi=1,\ \cos n\pi=\cos 4r\pi=1.$$

$$\therefore\ a_n=0\ ;\ \ \therefore\ a_4=a_8=a_{12}=\ldots=0.$$

n even but not multiple of four $=4r+2$ say.

$$\cos\frac{n\pi}{2}=\cos(2r+1)\pi=-1 \text{ and } \cos n\pi=1.$$

$$\therefore\ a_n=\frac{2kl}{n^2\pi^2}[2(-1)-1-1]=-\frac{8kl}{n^2\pi^2} \text{ for } n=2,6,10\ldots$$

$$\therefore\ a_2=\frac{-8kl}{\pi^2}\cdot\frac{1}{2^2},\ a_6=\frac{-8kl}{\pi^2}\cdot\frac{1}{6^2},\ a_{10}=\frac{-8kl}{\pi^2}\cdot\frac{1}{10^2}\cdots$$

$$\therefore\ f(x)=\frac{\lambda l}{4}-\frac{8\lambda l}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\frac{1}{10^2}\cos\frac{10\pi x}{l}+\dots\right.$$

Ex. 41. *If* $f(x)=\frac{l}{4}-x$, *when* $0<x<\frac{l}{2}$

$=x-\frac{3}{4}l$ *when* $l/2<x<l$,

prove that

$$f(x)=\frac{2l}{\pi^2}\left(\cos\frac{2\pi x}{l}+\frac{1}{3^2}\cos\frac{6\pi x}{l}+\frac{1}{5^2}\cos\frac{10\pi x}{l}+\dots\right).$$

In the interval $(0, l)$, $f(x)$ when expanded in cosine series is $\quad a_0+\sum_1^{\infty} a_n\cos\frac{n\pi x}{l}$.

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}\left[\int_0^{l/2}\left(\frac{l}{4}-x\right)dx+\int_{l/2}^l\left(x-\tfrac{3}{4}l\right)dx\right]$$

$$=\frac{1}{l}\left[\left(\frac{l}{4}\cdot\frac{l}{2}-\frac{1}{2}\cdot\frac{l^2}{4}\right)+\left\{\frac{1}{2}\left(l^2-\frac{l^2}{4}\right)-\frac{3}{4}l\left(l-\frac{l}{2}\right)\right\}\right]$$

$$=\frac{1}{l}\cdot\frac{l^2}{8}[1-1+4-1-6+3]=0.$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^{l/2}\left(\frac{l}{4}-x\right)\cos\frac{n\pi x}{l}\,dx+\int_{l/2}^l\left(x-\tfrac{3}{4}l\right)\cos\frac{n\pi x}{l}\,dx\Bigg]$$

$$=\frac{2}{l}\left[\left\{\left(\frac{l}{4}-x\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(-1)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right.$$

$$\left.+\left\{\left(x-\tfrac{3}{4}l\right)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(1)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_{l/2}^l\right]$$

$$=\frac{2}{l}\left[\left\{-\frac{l}{4}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right\}\right.$$

$$\left.+\left\{\frac{l}{4}\cdot\frac{l}{n\pi}\sin n\pi+\frac{l}{4}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2}\right)+\frac{l^2}{n^2\pi^2}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right\}\right].$$

$$a_n=\frac{2}{l}\cdot\frac{l^2}{n^2\pi^2}\left[1+\cos n\pi-2\cos\frac{n\pi}{2}\right]$$

$$= -\frac{2l}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right].$$

Now arguing as in last question,

$$a_1 = a_3 = a_5 = \ldots = 0 \text{ and } a_4 = a_8 = a_{12} = 0.$$

$$a_2 = -\frac{2l}{\pi^2}\cdot\frac{-4}{2^2} = \frac{2l}{\pi^2},\ a_6 = -\frac{2l}{\pi^2}\cdot\frac{-4}{6^2} = \frac{2l}{\pi^2}\cdot\frac{1}{3^2}$$

$$a_{10} = -\frac{3l}{\pi^2}\cdot\frac{-4}{10^2} = \frac{2l}{\pi^2}\cdot\frac{1}{5^2} \text{ and so on.}$$

$$\therefore\quad f(x) = \frac{2l}{\pi^2}\left[\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{6\pi x}{l} + \frac{1}{5^2}\cos\frac{10\pi x}{l}\ldots\right].$$

Ex. 42. *Prove that* $\frac{l}{2} - x = \frac{l}{\pi}\sum_1^\infty \frac{l}{n}\sin\frac{2n\pi x}{l},\ 0 < x < l.$

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l},$$

where $\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(\frac{l}{2} - x\right).\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[\left(\frac{l}{2} - x\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[-\frac{l}{2}\cdot\frac{-l}{n\pi}\cos n\pi - \frac{l}{2}\cdot\frac{-l}{n\pi}.1\right]$$

$$= \frac{2}{l}\cdot\frac{l^2}{2n\pi}(\cos n\pi + 1) = \frac{l}{\pi n}.(1 + \cos n\pi).$$

n even. Then $\cos n\pi = 1$; $\therefore\ b_n = \frac{2l}{n\pi}$.

$$\therefore\quad b_2 = \frac{2l}{\pi}\cdot\frac{1}{2} = \frac{l}{\pi}.1\ ;\ b_4 = \frac{2l}{\pi}\cdot\frac{1}{4} = \frac{l}{\pi}\cdot\frac{1}{2}\ ;\ b_6 = \frac{l}{\pi}\cdot\frac{1}{3}\ldots$$

n odd. Then $\cos n\pi = -1$; $\therefore\ b_n = 0$

or $$b_1 = b_3 = b_5 = \ldots = 0.$$

$$\therefore \quad f(x)=b_1 \sin \frac{\pi x}{l}+b_2 \sin \frac{2\pi x}{l}+b_3 \sin \frac{3\pi x}{l} + b_4 \sin \frac{4\pi x}{l}+\dots$$

$$=\frac{l}{\pi}\left(\sin \frac{2\pi x}{l}+\frac{1}{2} \sin \frac{4\pi x}{l}+\frac{1}{3} \sin \frac{6\pi x}{l}\dots\right)$$

$$=\frac{l}{\pi}\sum_1^\infty \frac{1}{n} \sin \frac{2n\pi x}{l}.$$

Ex. 43. *Prove that* $\left(\frac{l}{2}-x\right)^2=\frac{l^2}{12}+\frac{l^2}{\pi^2}\sum_1^\infty \frac{1}{n^2} \cos \frac{2n\pi x}{l}$, $0 < x < l$.

Proceed exactly as above, taking

$$f(x)=a_0+\sum_1^\infty a_n \cos \frac{n\pi x}{l}.$$

Ex. 44. *Find a Fourier's series for a function of x which has the value c when x lies between 0 and a and the value 0 when x lies between a and l.*

(Agra 46, 48, 61 ; Rajputana 48)

$$f(x)=c, \quad 0 < x < a;$$
$$=0, \quad a < x < l.$$

Let $f(x)=a_0+\sum_1^\infty a_n \cos \frac{n\pi x}{l}$ in cosine series

$$a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}\left(\int_0^a c\,dx+\int_a^l 0\,dx\right)=\frac{ca}{l},$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos \frac{n\pi x}{l}\,dx=\frac{2}{l}\int_0^a c \cos \frac{n\pi x}{l}\,dx+0$$

$$=\frac{2c}{l}\cdot\frac{l}{n\pi}\left[\sin \frac{n\pi x}{l}\right]_0^a=\frac{2c}{n\pi}\sin \frac{n\pi a}{l}.$$

$$\therefore \quad a_1=\frac{2c}{\pi}\sin \frac{\pi a}{l},\ a_2=\frac{2c}{2\pi}\sin \frac{2\pi a}{l},\dots$$

$$\therefore \quad f(x)=\frac{ca}{l}+\frac{2c}{\pi}\left(\sin\frac{\pi a}{l}\cos\frac{\pi x}{l}+\frac{1}{2}\sin\frac{2\pi a}{l}\cos\frac{2\pi x}{l}+\ldots\right).$$

Ex. 45. *A function $f(x)$ is defined as under :*

$$f(x)=x^2, \qquad 0<x<l/2\ ;$$
$$=0, \qquad l/2<x<l.$$

Express the function by means of series of sines and also by means of a series of multiples of $\pi x/l$.

Sine Series. $f(x)=\sum_1^\infty b_n \sin\frac{n\pi x}{l}$,

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}=\frac{2}{l}\left[\int_0^{l/2} x^2\sin\frac{n\pi x}{l}+\int_{l/2}^l 0\,dx\right.$$

$$=\frac{2}{l}\left[x^3\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right.$$
$$\left.+2.\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$=\frac{2}{l}\left[\frac{l^2}{4}.\frac{-l}{n\pi}\cos\frac{n\pi}{2}+2.\frac{l}{2}.\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+2.\frac{l^3}{n^3\pi^3}.\left(\cos\frac{n\pi}{2}-1\right)\right].$$

Now give n the numerical values 1, 2, 3, and find b_1, b_2, b_3.

Cosine series. *Similarly write*

$$f(x)=a_0+\sum_1^\infty a_n\cos\frac{n\pi x}{l},$$

where
$$a_0=\frac{1}{l}\int_0^l f(x)\,dx=\frac{1}{l}\left[\int_0^{l/2} x^2\,dx+\int_{l/2}^l 0\,dx\right]$$
$$=\frac{1}{l}.\frac{1}{3}\left(\frac{l}{2}\right)^3=\frac{l^2}{24},$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{l}\left[\int_0^{l/2} x^2\cos\frac{n\pi x}{l}+\int_{l/2}^l 0\,dx\right].$$

Evaluate the integral as above and put $n=1, 2, 3$

Ex. 46. *Find a Fourier's Series for the function defined by the equations*

$$f(x)=-l \quad \text{for} \quad -l<x<0,$$
$$=0 \quad \text{at} \quad x=0$$
$$=l \quad \text{for} \quad 0<x<l.$$

Hence deduce that $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\ldots$

In the interval $(-l, l)$,

$$f(x)=a_0+\sum_1^{\infty}\left(a_n \cos \frac{n\pi x}{l}+b_n \sin \frac{n\pi x}{l}\right),$$

$$a_0=\frac{1}{2l}\int_{-l}^{l} f(x)\,dx=\frac{1}{2l}\left[\int_{-l}^{0} f(x)\,dx+\int_0^l f(x)\,dx\right]$$

$$=\frac{1}{2l}\left[\int_{-l}^{0}-1.dx+\int_0^l 1.dx\right]=\frac{1}{2l}(-l+l)=0.$$

$$a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}$$

$$=\frac{1}{l}\left[\int_{-l}^{0}-1.\cos\frac{n\pi x}{l}\,dx+\int_0^l 1.\cos\frac{n\pi x}{l}.dx\right]$$

$$=\frac{1}{l}\cdot\frac{l}{n\pi}\left[\left(-\sin\frac{n\pi x}{l}\right)_{-l}^{0}+\left(\sin\frac{n\pi x}{l}\right)_0^l\right]=0, \quad \because \sin n\pi=0$$

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}$$

$$=\frac{1}{l}\left[\int_{-l}^{0}-1.\sin\frac{n\pi x}{l}\,dx+\int_0^l 1.\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{1}{l}\cdot\frac{l}{n\pi}\left[\left(\cos\frac{n\pi x}{l}\right)_{-l}^{0}-\left(\cos\frac{n\pi x}{l}\right)_0^l\right]$$

$$=\frac{1}{n\pi}[(1-\cos n\pi)-(\cos n\pi-1)]$$

$$=\frac{2}{n\pi}(1-\cos n\pi).$$

If n be even, then $b_n=0$, *i.e.*, $b_2=b_4=b_6=\ldots=0$.

If n be odd, then $b_n=\frac{2}{n\pi}(1+1)=\frac{4}{n\pi}$.

$$\therefore \quad b_1=\frac{4}{\pi},\ b_3=\frac{4}{\pi}\cdot\frac{1}{3},\ b_5=\frac{4}{\pi}\cdot\frac{1}{5}\cdots$$

Now $a_0=0$ and $a_n=0$.

$$\therefore \quad f(x)=b_1 \sin\frac{\pi x}{l}+b_2 \sin\frac{2\pi x}{l}+b_3 \sin\frac{3\pi x}{l}$$

$$=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\ldots\right].$$

Putting $x=l/2$, then $f(x)=1$ for $0<x<l$.

$$\therefore \quad f(x)=1 \text{ for } x=l/2.$$

$$\therefore \quad 1=\frac{4}{\pi}\left[\sin\frac{\pi}{2}+\frac{1}{3}\sin\frac{3\pi}{2}+\frac{1}{5}\sin\frac{5\pi}{2}\ldots\right]$$

or

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

Ex. 47. *Show that a function defined as*

$$\begin{aligned} f(x) &= -l \quad \textit{for} \quad -2l<x<-l \\ &= -x \quad \textit{for} \quad -l<x<0 \\ &= x \quad \textit{for} \quad 0<x<l \\ &= l \quad \textit{for} \quad l<x<2l \end{aligned}$$

can be represented as

$$\frac{3l}{4}-\frac{2l}{\pi^2}\sum_0^\infty \frac{1}{(2m+1)^2}\left[2\cos\frac{(2m+1)\pi x}{2l}+\cos\frac{(2m+1)\pi x}{l}\right].$$

(Rajputana 57)

Note here that the interval is $-2l$ to $2l$. Procedure is same.

$$f(x)=a_0+\sum_1^\infty\left(a_n\cos\frac{n\pi x}{2l}+b_n\sin\frac{n\pi x}{2l}\right) \quad (\text{Note } 2l).$$

$$a_0=\frac{1}{2(2l)}\int_{-2l}^{2l} f(x)\,dx$$

$$=\frac{1}{4l}\left[\int_{-2l}^{-l} l\,dx+\int_{-l}^{0} -x\,dx+\int_0^l x\,dx+\int_l^{2l} l\,dx\right]$$

$$=\frac{1}{4l}\left[l(-l+2l)-\frac{1}{2}(0-l^2)+\frac{1}{2}(l^2-0)+l(2l-l)\right]$$

$$=\frac{1}{4l}\, l^2\left[1+\frac{1}{2}+\frac{1}{2}+1\right]=\frac{3}{4}l.$$

$$a_n=\frac{1}{(2l)}\int_{-2l}^{2l} f(x)\cos\frac{n\pi x}{2l}\,dx$$

$$=\frac{1}{2l}\left[\int_{-2l}^{-l} l\cos\frac{n\pi x}{2l}\,dx+\int_{-l}^{0}-x\cos\frac{n\pi x}{2l}\,dx\right.$$

$$\left.+\int_0^l x\cos\frac{n\pi x}{2l}\,dx+\int_l^{2l} l\cos\frac{n\pi x}{2l}\,dx\right]$$

$$=\frac{1}{2l}\left[l\,\frac{2l}{n\pi}\left\{\left(\sin\frac{n\pi x}{2l}\right)_{-2l}^{l}+\left(\sin\frac{n\pi x}{2l}\right)_l^{2l}\right\}\right.$$

$$\left\{x.\left(\frac{2l}{n\pi}\sin\frac{n\pi x}{2l}\right)-(1)\left(-\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{2l}\right)_{-l}^{0}\right.$$

$$\left.+\left\{x\left(\frac{2l}{n\pi}\sin\frac{n\pi x}{2l}\right)-(1)\left(-\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{2l}\right)\right\}_0^l\right]$$

$$=\frac{1}{2l}\left[\frac{2l^2}{n\pi}\left\{\left(\sin\frac{n\pi}{2}-0\right)+\left(0-\sin\frac{n\pi}{2}\right)\right\}\right.$$

$$-\left\{0-l.\frac{2l}{n\pi}\sin\left(-\frac{n\pi}{2}\right)\right\}+\frac{4l^2}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)$$

$$\left.+\left\{\left(l.\frac{2l}{n\pi}\sin\frac{n\pi}{2}-0\right)+\frac{4l^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right)\right\}\right]$$

$$=\frac{1}{2l}\cdot\frac{4l^2}{n^2\pi^2}\left(1-\cos\frac{n\pi}{2}\right)(-1-1)=\frac{4l}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-1\right).$$

When n is odd, then $\cos\frac{n\pi}{2}=0$; $\therefore\quad a_n=-\frac{4l}{n^2\pi^2}$.

$$\therefore\quad a_1=-\frac{4l}{\pi^2}\cdot\frac{1}{1^2},\ a_3=-\frac{4l}{\pi^2}\cdot\frac{1}{3^2},\ a_5=-\frac{4l}{\pi^2}\cdot\frac{1}{5^2}\cdots$$

Again $a_4=a_8=a_{12}=\ \ldots=0,\quad\because\quad\cos 2r\pi=1.$

$$a_2=\frac{4l}{2^2\pi^2}(-1-1)=-\frac{4l}{\pi^2}\cdot\frac{2}{2^2},\ a_6=\frac{4l}{6^2.\pi^2}.(-1-1)=-\frac{4l}{\pi^2}\cdot\frac{2}{6^2}$$

Again $\quad b_n=\frac{1}{2l}\int_{-2l}^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx.$

Proceeding exactly as for a_n, we can show that it is zero.

$$\therefore\ f(x)=a_0+a_1\cos\frac{\pi x}{2l}+a_2\cos\frac{2\pi x}{2l}+a_3\cos\frac{3\pi x}{2l}+a_4\cos\frac{4\pi x}{2l}+\ldots$$

$$=\frac{3l}{4}-\frac{4l}{\pi^2}\left[\cos\frac{\pi x}{2l}+\frac{2}{2^2}\cos\frac{2\pi x}{2l}+\frac{1}{3^2}\cos\frac{3\pi x}{2l}+\frac{1}{5^2}\cos\frac{5\pi x}{2l}-\frac{2}{6^2}\cos\frac{6\pi x}{2l}+\ldots\right]$$

$$=\frac{3l}{4}-\frac{4l}{\pi^2}\left[\left(\cos\frac{\pi x}{2l}+\frac{1}{3^2}\cos\frac{3\pi x}{2l}+\frac{1}{5^2}\cos\frac{5\pi x}{2l}+\ldots\right)+\frac{1}{2}\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\ldots\right)\right]$$

$$=\frac{3l}{4}-\frac{2l}{\pi^2}\left[2\left(\cos\frac{\pi x}{2l}+\frac{1}{3^2}\cos\frac{3\pi x}{2l}+\frac{1}{5^2}\cos\frac{5\pi x}{2l}+\ldots\right)+\left(\cos\frac{\pi x}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\ldots\right)\right].$$

Above can be put in the form

$$\frac{3l}{4}-\frac{2l}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\left\{2\cos\frac{(2m+1)\pi x}{2l}+\cos\frac{(2m+1)\pi x}{l}\right\}.$$

Proved.

Ex. 48. *Find a function $f(x)$ which shall be periodic with period $2l$ and shall be $=l/4$ from $-l$ to $-l/2$, $=x^2/l$ from $-l/2$ to $l/2$, $=l/4$ from $l/2$ to l.*

$$f(x)=a_0+\sum_1^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right).$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)\,dx.$$

$$=\frac{1}{2l}\left[\int_{-l}^{-l/2}\frac{l}{4}\,dx+\int_{-l/2}^{l/2}\frac{x^2}{l}\,dx+\int_{l/2}^{l}\frac{l}{4}\,dx\right]$$

$$=\frac{1}{2l}\left[\frac{l}{4}\left(-\frac{l}{2}+l\right)+\frac{1}{3l}\left(\frac{l^3}{8}+\frac{l^3}{8}\right)+\frac{l}{4}\left(l-\frac{l}{2}\right)\right]$$

$$=\frac{1}{2l}\cdot l^2\left[\frac{1}{8}+\frac{1}{12}+\frac{1}{8}\right]=\frac{1}{6}l$$

$$a_n=\frac{1}{l}\left[\int_{-l}^{-l/2}\frac{l}{4}\cos\frac{n\pi x}{l}\,dx+\int_{-l/2}^{l/2}\frac{x^2}{l}\cos\frac{n\pi x}{l}\,dx+\int_{l/2}^{l}\frac{l}{4}\cos\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{1}{l}\left[\frac{l}{4}\cdot\frac{l}{n\pi}\left\{\left(\sin\frac{n\pi x}{l}\right)_{-l}^{-l/2}+\left(\sin\frac{n\pi x}{l}\right)_{l/2}^{l}\right\}+\frac{2}{l}\left\{x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)-(2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)+2\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right]\quad\text{(Prop. V)}$$

$$=\frac{l}{4n\pi}\left\{\left(-\sin\frac{n\pi}{2}-0\right)+\left(0-\sin\frac{n\pi}{2}\right)\right\}+\frac{1}{l}\cdot\frac{2}{l}\left\{\frac{l^2}{4}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2}+l.\frac{l^2}{n^2\pi^2}.\cos\frac{n\pi}{2}-\frac{2l^3}{n^3\pi^3}\sin\frac{n\pi}{2}\right\}$$

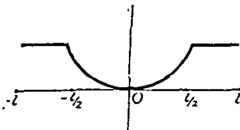
$$=-\frac{2l}{4n\pi}-\frac{2l}{4n\pi}\sin\frac{n\pi}{2}+\frac{2l}{n^2\pi^2}\cos\frac{n\pi}{2}-\frac{4l}{n^3\pi^3}\sin\frac{n\pi}{2}$$

$$=\frac{2l}{n^2\pi^2}\left[\cos\frac{n\pi}{2}-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right].$$

We can show as above that $b_n=0$; otherwise also, we can say that since when x be changed to $-x$ the function remains the same, hence we should expect cosine series only.

$$\therefore\quad f(x)=\tfrac{1}{6}l+\sum_1^\infty\frac{2l}{n^2\pi^2}\left(\cos\frac{n\pi}{2}-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{l}.$$

The graph of the function is composed of line $y=l/4$ from $-l$ to $-l/2$, parabola $y=x^2/l$ from $-l/2$ to $l/2$ and the line $y=l/4$ from $-l/2$ to l.



Ex. 49. *Show that the series*

$$\frac{v_1+v_2+v_3}{3}+\frac{l}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{n\pi}{3}\left\{2(v_3-v_1)\sin\frac{2n\pi}{3}\sin\frac{n\pi x}{l}+2(v_2-v_1-v_3)\cos\frac{n\pi x}{l}\right\}.$$

is equal to v_1 *when* $-l < x < -l/3$,

v_2 *when* $-l/3 < x < l/3$,

v_3 *when* $l/3 < x < l$,

where v_1, v_2 *and* v_3 *are constants.* **(Agra 63)**

As usual in the interval $(-l, l)$,

$$f(x) = a_0 + \sum_1^\infty \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right) \qquad \ldots(1)$$

$$\begin{aligned}
a_0 &= \frac{1}{2l} \int_{-l}^{l} f(x)\, dx \\
&= \frac{1}{2l} \left[\int_{-l}^{-l/3} v_1\, dx + \int_{-l/3}^{l/3} v_2\, dx + \int_{l/3}^{l} v_3\, dx \right] \\
&= \frac{1}{2l} \left[v_1 \left(-\frac{l}{3} + l\right) + v_2 \left(\frac{l}{3} + \frac{l}{3}\right) + v_3 \left(l - \frac{l}{3}\right) \right] \\
&= \frac{1}{2l} \cdot \frac{2l}{3} (v_1 + v_2 + v_3) = \frac{v_1 + v_2 + v_3}{3}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{l} \left[\int_{-l}^{-l/3} v_1 \cos \frac{n\pi x}{l}\, dx + \int_{-l/3}^{l/3} v_2 \cos \frac{n\pi x}{l}\, dx + \int_{l/3}^{l} v_3 \cos \frac{n\pi x}{l}\, dx \right] \\
&= \frac{1}{l} \cdot \frac{l}{n\pi} \left[v_1 \left\{ \sin \frac{n\pi x}{l} \right\}_{-l}^{-l/3} + v_2 \left\{ \sin \frac{n\pi x}{l} \right\}_{-l/3}^{l/3} + v_3 \left\{ \sin \frac{n\pi x}{l} \right\}_{l/3}^{l} \right] \\
&= \frac{1}{n\pi} \left[v_1 \left(-\sin \frac{n\pi}{3} - 0\right) + v_2 \left(\sin \frac{n\pi}{3} + \sin \frac{n\pi}{3}\right) + v_3 \left(0 - \sin \frac{n\pi}{3}\right) \right] \\
&= \frac{1}{n\pi} \sin \frac{n\pi}{3} [-v_1 + 2v_2 - v_3] = \frac{1}{n\pi} \sin \frac{n\pi}{3} [2v_2 - v_1 - v_3].
\end{aligned}$$

$$b_n = \frac{1}{l} \left[\int_{-l}^{-l/3} v_1 \sin \frac{n\pi x}{l}\, dx + \int_{-l/3}^{l/3} v_2 \sin \frac{n\pi x}{l}\, dx + \int_{l/3}^{l} v_3 \sin \frac{n\pi x}{l}\, dx \right]$$

$$=\frac{1}{l}\left(-\frac{l}{n\pi}\right)\left[v_1\left(\cos\frac{n\pi x}{l}\right)_{-l}^{-l/3}+v_2\left(\cos\frac{n\pi x}{l}\right)_{-l/3}^{l/3}\right.$$
$$\left.+v_3\left(\cos\frac{n\pi x}{l}\right)_{l/3}^{l}\right]$$
$$=-\frac{1}{n\pi}\left[v_1\left(\cos\frac{n\pi}{3}-\cos n\pi\right)\right.$$
$$\left.+v_2\ (0)+v_3\left(\cos n\pi-\cos\frac{n\pi}{3}\right)\right]$$
$$=\frac{1}{n\pi}\left(\cos\frac{n\pi}{3}-\cos n\pi\right)(v_3-v_1)$$
$$=\frac{v_3-v_1}{n\pi}.2\sin\frac{2n\pi}{3}\sin\frac{n\pi}{3}.$$

Now putting for a_0, a_n and b_n in (1), we get the required result.

CHAPTER VII

CONVERGENCE OF IMPROPER INTEGRALS

§ 1. Definition.

Proper Integral. In the definite integral $\int_a^b f(x)\,dx$, when integral is bounded and the range of integration is finite, it is said to be proper integral.

Improper Integral. If however the range of integration is infinite and the integrand bounded, then it is called improper integral of first kind. It is also called infinite integral, *e.g.*

$$\int_0^\infty \frac{dx}{1+x^2}.$$

When the range of integration is finite and the integrand is unbounded, *i.e.* the integrand becomes infinite for some value in the range of integration, then it is said to be Improper integral of second kind, *e.g.*,

$$\int_1^2 \frac{dx}{(1-x)\,(2-x)} \quad \text{or} \quad \int_0^1 \frac{dx}{x^2}. \qquad \textbf{(Punjab 54)}$$

§ 2. Convergence of improper integrals of first kind, i.e. range of integration infinite and the integrand bounded.

Case. I. $\int_a^\infty f(x)\,dx.$

If $f(x)$ is integrable and bounded for all values of $x \geq a$, then

$$\int_a^\infty f(x)\,dx = \underset{x\to\infty}{\text{Lt}} \int_a^x f(x)\,dx. \qquad \ldots(1)$$

If the limit on the R.H.S. of (1) exists and is unique and finite, then the integral on L.H.S. is said to be convergent,

but if the limit is $+\infty$ or $-\infty$, then it is said to be divergent. If however the integral neither converges nor diverges, then it is said to **oscillate.**

Consider the integral $\int_0^\infty e^{x/2}\,dx$ Here the integral is bounded for all values of $x \geqslant 0$

$$\int_0^\infty e^{x/2}\,dx = \underset{x\to\infty}{\text{Lt}} \int_0^x e^{x/2}\,dx = \underset{x\to\infty}{\text{Lt}}\ 2\left[e^{x/2}\right]_0^x$$

$$= \underset{x\to\infty}{\text{Lt}}\ 2\,[e^{x/2}-1] = \infty$$

Since the above limit is ∞, hence the given integral is divergent.

If however we consider the integral $\int_0^\infty e^{-x}\,dx$, then

$$\int_0^\infty e^{-x}\,dx = \underset{x\to\infty}{\text{Lt}} \int_0^x e^{-x}\,dx = \underset{x\to\infty}{\text{Lt}} \left[-e^{-x}\right]_0^x$$

$$= \underset{x\to\infty}{\text{Lt}}\ [1-e^{-x}] = 1\ ;$$

$$\because\ \underset{x\to\infty}{\text{Lt}}\ e^{-x} = \underset{x\to\infty}{\text{Lt}}\ \frac{1}{e^x} = \frac{1}{\infty} = 0.$$

Since the limits exists and is finite and unique, hence the given integral is convergent

Similarly, we can show that $\int_0^\infty \sin x\,dx$ oscillates.

$$\because\ \int_0^\infty \sin x\,dx = \underset{x\to\infty}{\text{Lt}} \int_0^x \sin x\,dx = \underset{x\to\infty}{\text{Lt}} \left[-\cos x\right]_0^x$$

$$= \underset{x\to\infty}{\text{Lt}}\ [1-\cos x].$$

The above limit does not exist and hence the function neither converges nor diverges. Therefore it oscillates.

Case II. $\int_{-\infty}^{b} f(x)\, dx.$

If $f(x)$ is integrable and bounded for all values of x in the interval (a, b), where $b > a$, then

$$\int_{-\infty}^{b} f(x)\, dx = \underset{x \to -\infty}{\text{Lt}} \int_{x}^{b} f(x)\, dx. \qquad ..(2)$$

If the limit of R H.S. of (2) exists and is unique and finite, then the integral on L.H.S. of (2) is said to be convergent but if the limit is $+\infty$ or $-\infty$, then it is said to be divergent.

Consider the integral $\int_{-\infty}^{0} \sinh x\, dx$.

$$I = \frac{1}{2}\int_{-\infty}^{0} (e^{x} - e^{-x})\, dx = \underset{x \to -\infty}{\text{Lt}} \frac{1}{2}\int_{x}^{0} e^{x}\, dx - \underset{x \to -\infty}{\text{Lt}} \frac{1}{2}\int_{x}^{0} e^{-x}\, dx$$

$$= \underset{x \to -\infty}{\text{Lt}} \tfrac{1}{2}[1 - e^{x}] - \underset{x \to -\infty}{\text{Lt}} \tfrac{1}{2}[e^{-x} - 1]$$

$$= \tfrac{1}{2}[1 - 0] - \tfrac{1}{2}[\infty - 1] = -\infty.$$

Hence the integral diverges to $-\infty$.

From above it follows clearly that integral $\int_{-\infty}^{0} e^{x}\, dx$ converges to 1, whereas $\int_{-\infty}^{0} e^{-x}\, dx$, diverges to $+\infty$. Similarly, we can say that $\int_{-\infty}^{0} \cosh x\, dx$ diverges to $+\infty$.

Case III. $\int_{-\infty}^{\infty} f(x)\, dx.$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_{a}^{\infty} f(x)\, dx. \qquad ...(3)$$

The integral on L.H.S. of (3) will converge *if both the infinite integrals on the R.H.S. are convergent and it is then equal to their sum.*

Consider the integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}\, dx$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}\, dx = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \quad \ldots (4)$$

$$= \underset{x \to -\infty}{\text{Lt}} \int_{x}^{0} \frac{dx}{1+x^2} + \underset{x \to \infty}{\text{Lt}} \int_{0}^{x} \frac{dx}{1+x^2}$$

$$= \underset{x \to -\infty}{\text{Lt}} \left[\tan^{-1} x\right]_{x}^{0} + \underset{x \to \infty}{\text{Lt}} \left[\tan^{-1} x\right]_{0}^{x}$$

$$= \underset{x \to -\infty}{\text{Lt}} [0 - \tan^{-1} x]$$

$$+ \underset{x \to \infty}{\text{Lt}} [\tan^{-1} x - 0]$$

$$= -\tan^{-1}(-\infty) + \tan^{-1} \infty$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Above shows that both the integrals on R.H.S. of (4) converge and hence the given integral also converges to their sum π.

§ 3. **Necessary and sufficient condition for the convergence of $\int_{a}^{\infty} f(x)\, dx$.**

The integral $\int_{a}^{\infty} f(x)\, dx$ is said to converge to the value I, when corresponding to any positive number ϵ, however small there exists a +ive number m, such that

$$\left| I - \int_{a}^{x} f(x)\, dx \right| < \epsilon \quad \textit{for all values of } x \geqslant m.$$

Also a necessary and sufficient condition for the convergence of the integral $\int_{a}^{\infty} f(x)\, dx$ is that, given ϵ however small,

a number m can be found such that $\left|\int_{x_1}^{x_2} f(x)\,dx\right| < \epsilon$ *for all values of* x_1 *and* x_2 *for which* $x_2 > x_1 > m$.

§ 4. Tests for convergence of $\int_a^\infty f(x)\,dx$.

1. Comparison Test. *Let* $f(x)$ *and* $\phi(x)$ *be two functions which are bounded and integrable in the interval* (a, ∞). *Also let* $f(x) < \phi(x)$ *when* $x > a$ *; then* $\int_a^\infty f(x)\,dx$ *is convergent if* $\int_a^\infty \phi(x)\,dx$ *is convergent and* $\int_a^\infty f(x)\,dx \leq \int_a^\infty \phi(x)\,dx$. *In case* $f(x) > \phi(x)$ *and* $\int_a^\infty \phi(x)\,dx$ *diverges, then* $\int_a^\infty f(x)\,dx$ *also diverges.*

Note. The above test corresponds to the comparison test for convergency and divergency of series which the students must have read in author's book on algebra for degree classes

Auxiliary Integral.

$\int_a^\infty \frac{dx}{x^n}$ where $a > 0$ is convergent when $n > 1$ and divergent when $n \leq 1$. (Agra 50)

$$\int_a^\infty \frac{dx}{x^n} = \underset{X\to\infty}{\text{Lt}} \int_a^X x^{-n}\,dx = \underset{X\to\infty}{\text{Lt}} \left[\frac{x^{1-n}}{-(n-1)}\right]_a^X$$

$$= \underset{X\to\infty}{\text{Lt}} \frac{1}{n-1}\left[a^{1-n} - X^{1-n}\right].$$

If $n > 1$, then $1-n$ is $-$ive and $\underset{X\to\infty}{\text{Lt}} X^{1-n} = \frac{1}{\infty} = 0$.

$$\therefore \int_a^\infty \frac{dx}{x^n} = \frac{a^{1-n}}{n-1} \text{ if } n > 1.$$

If $n < 1$, then $1-n$ is $+$ive and $\underset{X\to\infty}{\text{Lt}} X^{1-n}$ is ∞

$$\therefore \quad \int_a^\infty \frac{dx}{x^n} = \frac{1}{-\text{ive}} [a^{1-n} - \infty] = +\infty .$$

Hence the above integral diverges.

If n = 1, then

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x} = \underset{x\to\infty}{\text{Lt}} \int_a^x \frac{dx}{x}$$

$$= \underset{x\to\infty}{Lt} (\log x - \log a) = \infty . \quad \text{Hence divergent.}$$

We shall use the above integral as auxiliary integral for the application of comparison test.

Ex. 1. *Test the convergence of integral* $\int_a^\infty \frac{\sin^2 x}{x^2} dx.$

(Punjab 52)

Let us choose $\phi(x) = \frac{1}{x^2}$ and since $\sin x < 1$,

$$\therefore \quad f(x) < \phi(x)$$

Hence $\int_a^\infty \frac{\sin^2 x}{x^2} dx$ converges if $\int_a^\infty \frac{1}{x^2} dx,\ a > 0$ converges.

But we know that $\int_a^\infty \frac{1}{x^2} dx$ converges because $n=2$ *i.e.* > 1 ; otherwise also

$$\int_a^\infty \frac{1}{x^2} dx = \underset{x\to\infty}{\text{Lt}} \int_a^x \frac{1}{x^2} dx = \underset{x\to\infty}{\text{Lt}} \left(\frac{1}{a} - \frac{1}{x}\right) = \frac{1}{a} \ i.e. \text{ convergent.}$$

Hence by comparison test, the given integral is also convergent.

Ex. 2. *Test the convergence of* $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$.

Here let us choose $\phi(x) = \frac{1}{x}$; $\therefore \ f(x) = \frac{1}{\sqrt{(x^2-1)}}$.

$$\sqrt{(x^2-1)} < \sqrt{(x^2)} \quad \text{or} \quad \frac{1}{\sqrt{(x^2-1)}} > \frac{1}{x} \ i.e. f(x) > \phi(x)$$

when $x > 2$.

Hence $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$ diverges according as $\int_2^\infty \frac{dx}{x}$ diverges

But $\int_2^\infty \frac{dx}{x}$ $(n=1)$ diverges Hence by comparison test, the given integral also diverges.

Ex. 3. *Test the convergence of* $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$.

Here let us choose $\phi(x)=\frac{1}{x^2}$ *and* $f(x)=\frac{1}{x\sqrt{(1+x^2)}}$.

Clearly $f(x) < \phi(x)$; but $\int_a^\infty \phi(x)\,dx=\int_a^\infty \frac{1}{x^2}\,dx,\quad n=2,$ *i.e.* > 1, converges and hence by comparison test, the given integral also converges.

Ex. 4. *Test the convergence of integral* $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$.

Here let us choose $\phi(x)=\frac{1}{1+x^2}$ and $f(x)=\frac{\cos x}{1+x^2}$.

Since $\cos x < 1$, $\quad\therefore\ f(x) < \phi(x)$ and hence by comparison test $\int_0^\infty f(x)\,dx$ will converge if $\int_0^\infty \phi(x)\,dx$ converges.

$$\text{Now} \int_0^\infty \phi(x)\,dx=\underset{x\to\infty}{\text{Lt}}\int_0^x \frac{1}{1+x^2}\,dx=\underset{x\to\infty}{\text{Lt}}\tan^{-1}x=\frac{\pi}{2}.$$

Hence $\int_0^\infty \phi(x)\,dx$ converges to $\frac{\pi}{2}$ and so does the $\int_0^\infty f(x)\,dx$ converge to $\frac{\pi}{2}$.

Ex. 5. *Show that* $\int_0^\infty e^{-x^2}\,dx$ *is convergent.*

(Sagar 62 ; Punjab 56)

$$\int_0^\infty e^{-x^2}\,dx=\int_0^1 e^{-x^2}\,dx+\int_1^\infty e^{-x^2}\,dx.$$

The first integral is bounded and the range of integration is finite. Hence it is a proper integral and therefore convergent.

Now for values of $x > 1$,

$$e^{-x^2} < xe^{-x^2}, \quad i.e. \quad f(x) < \phi(x)$$

and hence $\int_1^\infty f(x)\,dx$ will be convergent if $\int_1^\infty \phi(x)\,dx$ is convergent.

i.e. $\underset{X\to\infty}{\text{Lt}} \int_1^X xe^{-x^2}\,dx$ is finite

or $\underset{X\to\infty}{\text{Lt}} -\frac{1}{2}\left[e^{-x^2}\right]_1^X = \underset{X\to\infty}{\text{Lt}} \frac{1}{2}\left[\frac{1}{e}-\frac{1}{e^{X^2}}\right] = \frac{1}{2e}$, *i.e. finite.*

Hence $\int_1^\infty e^{-x^2}\,dx$ is also convergent.

Therefore the given integral being the sum of two convergent integrals is also convergent.

II Test. The μ-test.

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$. *If there is a +ive number* $\mu > 1$, *such that* $\underset{X\to\infty}{\text{Lt}} x^\mu f(x)$ *exists finitely the limit being neither zero nor infinite, then the integral* $\int_a^\infty f(x)\,dx$ *converges*

If there is a number $\mu \leqslant 1$, *such that* $\underset{X\to\infty}{\text{Lt}} x^\mu f(x)$ *exists and is not zero, then the integral* $\int_a^\infty f(x)\,dx$ *is divergent.*

Note. The value of μ is easily taken to be as under: "highest power of x in D^r — highest power of x in N^r."

Examine the convergence of the following integrals.

Ex. 1. $\int_0^\infty \frac{x\,dx}{(1-x)^3}$, $\mu=3-1=2$. **(Agra 53)**

$$\underset{x\to\infty}{\text{Lt}}\, x^\mu f(x)=\underset{x\to 0}{\text{Lt}}\, x^2 \frac{x}{(1-x)^3}=1, \quad \textit{i.e. finite.}$$

Since $\mu=2$, *i.e* >1, hence the given integral is convergent by μ-test.

Ex. 2. $\int_1^\infty \frac{dx}{x^{1/3}\,(1+x^{1/2})}$, $\mu=(\frac{1}{3}+\frac{1}{2})-0=\frac{5}{6}$. **(Agra 53)**

$$\underset{x\to\infty}{\text{Lt}}\, x^\mu f(x)=\underset{x\to\infty}{\text{Lt}}\, x^{5/6} \frac{1}{x^{1/3}\,(1+x^{1/2})}=1, \text{ i.e. finite.}$$

Since $\mu=\frac{5}{6}$, *i.e.* <1, hence the given integral is divergent by μ-test.

Ex. 3. $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$, *Convergent* $\mu=2$, *i.e.* >1.

Ex. 4. $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$. *Convergent.* $\mu=\frac{3}{2}$, *i.e.* >1.

Ex. 5. $\int_0^\infty \frac{x^3\,dx}{(a^2+x^2)^2}$. *Divergent.* $\mu=1$. **(Agra 55)**

Ex. 6 $\int_0^\infty \frac{x^{3/2}}{b^2x^2+c^2}\,dx$. *Divergent.* $\mu=\frac{1}{2}$, *i.e.* <1

(Rajputana 62)

Ex. 7. $\int_0^\infty \frac{x^{3/2}\,dx}{\sqrt{(x^4-a^4)}}\,dx$. Divergent. $\mu=2-\frac{3}{2}=\frac{1}{2}$, *i.e.* <1. **(Agra 50)**

Ex. 8. $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$, *m and n being +ive integers.*

(Karnatak 63 ; Agra 46, 63, 66)

$$\text{Now} \int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx=\int_0^a \frac{x^{2m}}{1+x^{2n}}\,dx+\int_a^\infty \frac{x^{2m}}{1+x^{2n}}\,dx.$$

Clearly the first integral being proper is convergent. *For the second integral choose*

$$\mu=2n-2m=2\,(n-m),$$

$$\underset{x\to\infty}{\text{Lt}}\ x^{\mu}\,\frac{x^{2m}}{1+x^{2n}}=\underset{x\to\infty}{\text{Lt}}\ \frac{x^{2n-2m+2m}}{1+x^{2n}}=\underset{x\to 0}{\text{Lt}}\ \frac{x^{2n}}{1+x^{2n}}=1$$

i.e. finite.

Now $\mu=2(n-m)$ and if $n>m$, then μ is >1

$\because$ n and m are +ive integers.

$\mu=2(n-m)$ and if $n\leqslant m$, then μ is <1.

Hence $\int_a^{\infty}\frac{x^{2m}}{1+x^{2n}}\,dx$ is convergent if $n>m$ and divergent if $n\leqslant m$.

Therefore $\int_0^{\infty}\frac{x^{2m}}{1+x^{2n}}$ is also convergent if $n>m$ and divergent if $n\leqslant m$

Ex. 9. $\int_a^{\infty}\frac{dx}{x\,(\log x)^{\mu+1}}$ *where* $a>1$. **(Agra 61)**

Put $\log x=t$, $\therefore$ $\frac{dx}{x}=dt$, and adjust the limits.

$$\therefore\quad \int_a^{\infty}\frac{dx}{x\,(\log x)^{\mu+1}}=\int_{\log a}^{\infty}\frac{dt}{t^{\mu+1}},$$

where $\log a$ is >0 as $a>1$.

Here $\mu=\mu+1$ and $\underset{\lambda\to\infty}{\text{Lt}}\ \frac{t^{\mu+1}}{t^{\mu+1}}=1$, finite

Hence the given integral converges, if $\mu+1>1$ *i.e.* $\mu>0$, *i.e.* +ive and divergent if $\mu+1\leqslant 1$ or $\mu\leqslant 0$, *i.e.* $\mu=0$ or $-$ive

III Test. Abel's Test for the convergence of integral of a product.

If $\int_a^{\infty}f(x)\,dx$ *is convergent and* $\phi(x)$ *is monotonic and bounded for* $x>a$, *then integral* $\int_a^{\infty}f(x)\,\phi(x)\,dx$ *is also convergent.*

Ex. 1. *Test the convergence of* $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\,dx$.

Let $\qquad f(x)=\frac{\cos x}{x^2}$ and $\phi(x)=(1-e^{-x})$.

Now since $\cos x < 1$, $\therefore$ $\frac{\cos x}{x^2} \leqslant \frac{1}{x^2}$ and $\int_a^\infty \frac{1}{x^2}\,dx$ is convergent because $n=2$ by § 4 P. 301.

Otherwise also, $\int_a^\infty \frac{1}{x^2}\,dx = \underset{x\to\infty}{\text{Lt}} \left[-\frac{1}{x}\right]_a^\infty = \frac{1}{a}$.

Hence $\int_a^\infty \frac{\cos x}{x^2}$ is convergent

Also $(1-e^{-x})$ is monotonic increasing and bounded function for $x > a$.

Therefore by Abel's Test, $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\,dx$ is convergent.

Ex. 2. *Test the convergence of* $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\,dx$.

Here $\qquad f(x)=\frac{\sin x}{x^2}$ and $\phi(x)=e^{-x}$.

Since $\sin x < 1$, $\therefore$ $\frac{\sin x}{x^2} < \frac{1}{x^2}$ and $\int_a^\infty \frac{1}{x^2}\,dx$ is convergent as in last example. Hence $\int_a^\infty \frac{\sin x}{x^2}\,dx$ is also convergent.

Again e^{-x} is monotonic increasing and bounded function for $x > a$.

Hence by Abel's Test, $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\,dx$ is convergent.

IV Test. Dirichlet's Test.

If $f(x)$ be bounded and monotonic and if $\underset{x\to\infty}{\text{Lt}} f(x)=0$,

then the integral $\int_a^\infty f(x)\,\phi(x)\,dx$ converges provided that $\left|\int_a^x \phi(x)\,dx\right|$ is bounded as x takes all finite values

Ex. 1. *Test the convergence of* $\int_a^\infty \frac{\sin x}{x^n}\,dx,\ n > 0.$

(Nagpur 50)

Here $f(x)=\frac{1}{x^n}$ and $\phi(x)=\sin x$

Clearly $\frac{1}{x^n}$ is monotonic and $\underset{x\to\infty}{\text{Lt}} \frac{1}{x^n}=0$, if $n > 0$.

Also $\left|\int_a^x \phi(x)\,dx\right|=\left|\int_a^x \sin x\,dx\right|=|\cos a-\cos x| \leqslant 2$,

so that $\int_a^x \sin x\,dx$ oscillates finitely, *i.e.* it is bounded.

Hence by Dirichlet's Test, $\int_a^\infty \frac{\sin x}{x^n}$ is convergent.

Ex. 2. *Test the convergence of* $\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx$.

Proceed exactly as in Ex. 1. Convergent.

Ex. 3. *Test the convergence of* $\int_0^\infty \frac{\sin x}{x}\,dx$.

(Karnatak 62)

$$\int_0^\infty \frac{\sin x}{x}\,dx=\int_0^1 \frac{\sin x}{x}\,dx+\int_1^\infty \frac{\sin x}{x}\,dx. \qquad \ldots(1)$$

$\underset{x\to 0}{\text{Lt}} \frac{\sin x}{x}\left(\text{form } \frac{0}{0}\right)=\underset{x\to 0}{\text{Lt}} \frac{\cos x}{1}=1$, showing thereby that $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral, *i.e.* convergent, and hence we shall proceed with the convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$.

Here $f(x)=\frac{1}{x}$ and $\phi(x)=\sin x$.

Clearly $\frac{1}{x}$ is monotonic and $\underset{x\to\infty}{\text{Lt}} \frac{1}{x}=0$

Also $\left|\int_a^x \phi(x)\,dx\right|=\left|\int_1^x \sin x\,dx\right|=|\cos 1-\cos x| \leqslant 2$,

so that $\int_1^x \sin x\,dx$ oscillates finitely, *i.e.* it is bounded

Hence by Dirichlet's Test, $\int_1^\infty \frac{\sin x}{x}\,dx$ is convergent and from (1), we conclude that $\int_0^1 \frac{\sin x}{x}\,dx$ is also convergent

Ex. 4. *Prove that the integral* $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx,\ a \geqslant 0,$ *is convergent* **(Rajputana 62 ; Punjab 54)**

$$\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx=\int_0^a e^{-ax}\frac{\sin x}{x}\,dx+\int_a^\infty e^{-ax}\frac{\sin x}{x}\,dx. \quad \ldots(1)$$

Now

$$\underset{x\to 0}{\text{Lt}}\ e^{-ax}\frac{\sin x}{x}\left(\text{form } \frac{0}{0}\right)=\underset{x\to 0}{\text{Lt}}\ \frac{e^{-ax}(\cos x-a\sin x)}{1}=1,$$

showing thereby that $\int_0^a e^{-ax}\frac{\sin x}{x}\,dx$ is a proper integral and hence convergent.

We will therefore proceed with the convergence of

$$\int_a^\infty e^{-ax}\frac{\sin x}{x}\,dx.$$

Let $\qquad f(x)=\frac{e^{-ax}}{x}$ and $\phi(x)=\sin x$.

Clearly $f(x)$ is bounded and monotonic decreasing function of x for all values of $x>0$.

Also $\qquad \underset{x\to\infty}{\text{Lt}}\ \frac{e^{-ax}}{x}=\underset{x\to\infty}{\text{Lt}}\ \frac{1}{xe^{ax}}=\frac{1}{\infty}=0.$

Now $\left|\int_a^x \phi(x)\,dx\right|=\left|\int_a^x \sin x\,dx\right|=|\cos a-\cos x| \leqslant 2$,

so that $\int_a^x \sin x\, dx$ oscillates finitely, *i.e.* it is bounded.

Hence by Dirichlet's Test, $\int_a^\infty e^{-ax} \frac{\sin x}{x} dx$ is convergent.

Therefore by (1), $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$ is also convergent.

Ex. 5. *Prove that* $\int_1^\infty \sin x^2\, dx$ *is convergent.*

(Sagar 63)

$$\int_1^\infty \sin x^2\, dx = \int_1^\infty \frac{1}{2x} . 2x \sin x^2\, dx.$$

Here $f(x) = \frac{1}{2x}$ and $\phi(x) = 2x \sin x^2$

Clearly $f(x)$ is bounded and monotonic decreasing function and $\underset{x\to\infty}{\text{Lt}} f(x) = \underset{x\to\infty}{\text{Lt}} \frac{1}{2x} = 0.$

Also

$$\left| \int_1^x \phi(x)\, dx \right| = \left| \int_1^x 2x \sin x^2\, dx \right| = |\cos 1^2 - \cos x^2| < 2.$$

Hence $\int_1^x 2x \sin x^2\, dx$ oscillated finitely, *i.e.* it is bounded.

Therefore by Dirichlet's Test, $\int_1^\infty \frac{1}{2x} . 2x \sin x^2\, dx$ is convergent or $\int_1^\infty \sin x^2\, dx$ is convergent.

Ex. 6. *Show that* $\int_1^\infty \frac{x}{1+x^2} \sin x\, dx$ *converges.*

$$f(x) = \frac{x}{1+x^2} \text{ and } \phi(x) = \sin x.$$

$f(x)$ is monotonic and $\underset{x\to\infty}{\text{Lt}} \frac{x}{1+x^2} = 0.$

Ex. 1. *Show that* $\int_1^\infty \frac{\sin x}{x^4}\,dx$ *is absolutely convergent.*

$$\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx = \underset{x\to\infty}{\text{Lt}} \int_1^x \left|\frac{\sin x}{x^4}\right| dx$$

$$\leq \underset{x\to\infty}{\text{Lt}} \frac{dx}{x^4} \quad \because \sin x \leq 1$$

$$\underset{x\to\infty}{\text{Lt}} \left[\frac{1}{3} - \frac{1}{3x^3}\right] = \frac{1}{3}.$$

$$\therefore \int_1^\infty \left|\frac{\sin x}{x^4}\right| dx \text{ is convergent.}$$

Hence $\int_1^\infty \frac{\sin x}{x^4}\,dx$ is absolutely convergent.

Ex. 2. *Show that* $\int_0^\infty e^{-a^2x^2} \cos bx\,dx$ *is absolutely convergent.*

$$\int_0^\infty |e^{-a^2x^2} \cos bx|\,dx < \int_0^\infty |e^{-a^2x^2}|\,dx \text{ as } \cos bx \leq 1.$$

Now by Ex. 5 P. 303, $\int_0^\infty e^{-a^2x^2}\,dx$ is convergent.

Hence the integral $\int_0^\infty e^{-a^2x^2} \cos bx\,dx$ is absolutely convergent.

Ex. 3. *Test the integral* $\int_0^\infty f(x)\,dx$ *for absolute convergence, where* $f(x)$ *is defined as follows :*

$f(x) = 1$ *for* $0 < x \leq 1$

$= 0$ *for* $n-1 < x \leq n - \frac{1}{n}$

$= (-1)^{n+1}$ *for* $n - \frac{1}{n} < x \leq n$, *where* $n = 2, 3, 4 \ldots$

$$\int_0^\infty f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^{1\frac{1}{2}} f(x)\,dx+\int_{1\frac{1}{2}}^2 f(x)\,dx+\ldots$$

For $n=2$, $f(x)=1$ for 1st integral, $f(x)=0$ for 2nd, and $f(x)=(-1)^{2+1}=-1$ for third

$$\therefore \quad \int_0^\infty f(x)\,dx=\int_0^1 1\,dx+\int_1^{1\frac{1}{2}} 0\,dx+\int_{1\frac{1}{2}}^2 (-1)\,dx+\ldots$$
$$=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}\ .$$

The above series is convergent and its sum is log 2 and hence the given integral is convergent.

$$\text{Now } \int_0^\infty |f(x)|\,dx=\int_0^1 |f(x)|\,dx+\int_1^{1\frac{1}{2}} |f(x)|\,dx$$
$$+\int_{1\frac{1}{2}}^2 |f(x)|\,dx+\ldots$$
$$=\int_0^1 1\,dx=\int_1^{1\frac{1}{2}} 0\,dx+\int_{1\frac{1}{2}}^2 (+1)\,dx$$
$$=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots$$

Above series $\Sigma \frac{1}{n^p}$, where $p=1$, is divergent

Hence $\int_0^\infty |f(x)|\,dx$ is divergent.

Thus we conclude that even though $\int_0^\infty f(x)\,dx$ converges, but not absolutely.

§ 6. Absolute convergence product.

If $f(x)$ is bounded when $x \geqslant a$ and integrable in the arbitrary interval (a, b) and $\int_a^\infty \phi(x)\,dx$ converges, then $\int_a^\infty \phi(x)\,f(x)\,dx$ is absolutely convergent.

Ex. 1. *Prove that $\int_a^\infty \frac{\sin mx}{a^2+x^2}\,dx$ and $\int_a^\infty \frac{\cos mx}{a^2+x^2}\,dx$ converge absolutely, when m and a are +ive.*

Let $f(x) = \cos mx$ or $\sin mx$ and they are bounded and integrable

Also $\int_a^\infty \left| \frac{1}{a^2+x^2} \right| dx = \int_a^\infty \frac{1}{a^2+x^2} dx$ is convergent by μ-test because $\mu = 2$ and $\underset{x \to \infty}{\text{Lt}} x^\mu \phi(x) = \text{Lt} \frac{x^2}{a^2+x^2} = 1$, *i.e.*, finite

Since $\mu > 1$, hence the above integral is convergent. Therefore $\int_a^\infty \phi(x)\, dx$ is absolutely convergent.

Hence $\int_a^\infty \frac{\sin mx}{a^2+x^2} dx$ and $\int_a^\infty \frac{\cos mx}{a^2+x^2} dx$ are absolutely convergent.

Ex. 2. $\int_a^\infty \frac{\sin x}{x^{1+n}} dx$ *and* $\int_a^\infty \frac{\cos x}{x^{1+n}} dx$ *are absolutely convergent, when n and a are +ive.*

Here $\int_a^\infty \frac{1}{x^{1+n}} dx$ is absolutely convergent by μ-test if $n+1 > 1$, *i.e.* n is +ive and $\sin x$ and $\cos x$ are bounded and integrable. Hence given integrals are absolutely convergent

§ 7. Convergence of improper integrals of second kind, *i.e.* when integrand is infinite and range of integration finite

$\int_a^b$ **f(x) dx, where f(x) is unbounded.**

Suppose $f(x)$ becomes unbounded at $x=a$, *i.e.* $x=a$ is the only point of infinite discontinuity in the interval (a, b) Therefore the integral $f(x)$ is bounded and integrable in the interval $(a+\epsilon, b)$, where $a < a+\epsilon < b$.

$\int_a^b f(x)\, dx = \underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b f(x)\, dx$, if the limit exists.

Similarly if f(x) becomes unbounded at x=b, then

$$\int_a^b \mathbf{f(x)\ dx} = \underset{\epsilon \to 0}{\text{Lt}} \int_a^{b-\epsilon} \mathbf{f(x)\ dx} \text{ if the limit exists.}$$

Again if $f(x)$ becomes unbounded at $x=a$ and $x=b$, *i.e.* when a and b are both points of infinite discontinuity, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b$

Both these integrals on R H S have one point of infinite discontinuity and have been discussed above

§ 8. Test for convergence of $\int_a^b \mathbf{f(x)\ dx}$.

I. Comparison Test.

If $\underset{x \to a}{Lt} \dfrac{f(x)}{\phi(x)}$=*finite, i.e. neither zero nor infinite, then the two integrals* $\int_a^b f(x)\,dx$ *and* $\int_a^b \phi(x)\,dx$ *are either both convergent or both divergent.*

Auxiliary Integral $\int_a^b \dfrac{dx}{(x-a)^n}$ is convergent when n < 1 and divergent if n ⩾ 1.

$x=c$ is the only point of infinite discontinuity.

$$\therefore \quad \int_a^b \frac{dx}{(x-a)^n} = \underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n}$$

$$= \underset{\epsilon \to 0}{\text{Lt}} \left[\frac{1}{-(n-1)\,(x-a)^{n-1}} \right]_{a+\epsilon}^b$$

$$= \underset{\epsilon \to 0}{\text{Lt}} \frac{1}{1-n} \left[(b-a)^{1-n} - \epsilon^{1-n} \right]$$

$= \dfrac{1}{1-n}(b-a)^{1-n}$ where n is < 1, *i.e.* $1-n$ is +ive

$= \infty$ when n is > 1, *i.e.* $(n-1)$ is +ive

or $1-n = -$ive and $\underset{\epsilon \to 0}{\text{Lt}}\ \epsilon^{1-n} - \infty$.

when $n=1$, $\int_a^b \frac{dx}{x-a} = \underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b \frac{dx}{x-a} = \underset{\epsilon \to 0}{\text{Lt}} \Big[\log(x-a)\Big]_{a+\epsilon}^b$

$$= \underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b [\log(b-a) - \log \epsilon] = \infty$$

From above we conclude that

$$\underset{\epsilon \to 0}{\text{Lt}} \int_{a+\epsilon}^b \frac{dx}{(x-a)^n} \text{ exists only when } n < 1.$$

Hence the given integral is convergent only when $n < 1$ and divergent when $n \geqslant 1$.

Note. If you compare this result with that of § 4 P. 301 you observe that results are opposite as in that case convergent when $n > 1$ and divergent when $n \leqslant 1$.

Ex. 1 (a) *Test the convergence of* $\int_0^1 \frac{dx}{x^3(1+x)^2}$.

Here the integrand $f(x)$ is unbounded at $x=0$

Let us choose $\phi(x) = \frac{1}{x^3} = \frac{1}{(x-0)^3}$ and we know that

$$\int_0^1 \frac{1}{(x-0)^3} dx \text{ is divergent as } n=3, \text{ i.e. } > 1.$$

Now $\underset{x \to 0}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x \to 0}{\text{Lt}} \frac{1}{x^3(1+x)^2} \cdot x^3 = \underset{x \to 0}{\text{Lt}} \frac{1}{(1+x)^2} = 1$,

i.e., finite.

Hence by comparison test both the integrals either converge or diverge together. Since $\int_0^1 \frac{1}{(x-0)^3} dx$ is divergent, hence the given integral is also divergent.

(b) $\int_0^1 \frac{dx}{x^2(1+x)^2}$ (Divergent). **(Sagar 64)**

Ex. 2. *Test the convergence of* $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$. **(Sagar 63)**

Here the integrand $f(x)$ is unbounded at $x=0$

Let us choose $\phi(x) = \frac{1}{x^{1/3}} = \frac{1}{(x-0)^{1/3}}$ and we know that

$\int_0^1 \frac{1}{(x-0)^{1/3}}\, dx$ is convergent as $n=\frac{1}{3}$, *i.e.* < 1

Now $\underset{x\to 0}{\text{Lt}} \frac{f(x)}{\phi(x)} = \underset{x\to 0}{\text{Lt}} \frac{1}{x^{1/3}(1+x^2)} x^{1/3} = \underset{x\to 0}{\text{Lt}} \frac{1}{1+x^2} = 1,$

i.e. finite.

Hence by comparison test both integrals either converge or diverge together

Since $\int_0^1 \frac{1}{x^{1/3}}\, dx$ is convergent, hence the given integral is also convergent

Ex. 3. *Prove that Beta function* $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *is convergent if m and n are both +ive*

(Vikram 62 ; Agra 64, 67 ; Karnatak 64)

1st Case. If $m-1$ and $n-1$ both are +ive, *i.e.* $m > 1$, $n > 1$, then the above integral is a proper integral and hence convergent.

2nd Case. If $m-1$ and $n-1$ both are +ive, *i.e.* $m < 1$ and $n < 1$, then 0 and 1 both are points of infinite discontinuity. If c be any point in the interval (0, 1), then

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^c x^{m-1}(1-x)^{n-1}\, dx + \int_c^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Convergence at 0. $\int_0^c x^{m-1}(1-x)^{n-1}\, dx$ has $x=0$ as a point of infinite discontinuity.

$I = \int_0^c \frac{(1-x)^{n-1}}{x^{1-m}}\, dx$, $m-1 = -$ive or $1-m = +$ive.

Choose $\phi(x) = \frac{1}{x^{1-m}}$ and $\int_0^c \frac{dx}{(x-0)^{1-m}}$ is convergent if $1-m < 1$ or $0 < m$ or $m > 0$, *i.e.* m +ive.

$$\underset{x\to 0}{\text{Lt}}\ \frac{f(x)}{\phi(x)}=\underset{x\to 0}{\text{Lt}}\ \frac{(1-x)^{n-1}}{x^{1-m}}\cdot x^{1-m}=1,\ i.e.\ \text{finite}.$$

Hence both the integrals converge or diverge together but since one of them converges, therefore the other is also convergent and the condition is that m is +ive.

Convergence at 1. $\int_c^1 x^{m-1}(1-x)^{n-1}\,dx$ has $x=1$ as a *point of infinite discontinuity because $n-1$ is $-$ive or $1-n$ is +ive.*

$$\therefore\quad \int_c^1 x^{m-1}(1-x)^{n-1}\,dx=\int_c^1 \frac{x^{m-1}}{(1-x)^{1-n}}\,dx.$$

Choose $\phi(x)=\dfrac{1}{(1-x)^{1-n}}$ and $\int_c^1 \dfrac{1}{(1-x)^{1-n}}\,dx$ is convergent if $1-n<1$ or $0<n$, *i.e.* $n>0$ or n is +ive.

$$\underset{x\to 1}{\text{Lt}}\ \frac{f(x)}{\phi(x)}=\frac{x^{m-1}}{(1-x)^{1-n}}\cdot(1-x)^{1-n}=1,\ i.e.\ \text{finite}.$$

Hence both the integrals converge or diverge together but since one of them converges, therefore the other is also convergent and the condition is that n is +ive. Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ being the sum of two convergent integrals is also convergent when both m and n are + ive.

Ex. 4. *Prove that $\int_0^{\pi/2} x^m \operatorname{cosec}^n x\,dx$ exists, if and only if $n<m+1$.* **(Delhi 52)**

$$I=\int_0^{\pi/2}\frac{x^m}{x^n}\left(\frac{x}{\sin x}\right)^n dx=\int_0^{\pi/2} x^{m-n}\left(\frac{x}{\sin x}\right)^n dx.$$

Now when $m-n>0$, *i.e.* +ive, then above integral is proper because $\underset{x\to 0}{\text{Lt}}\ \dfrac{x}{\sin x}=\underset{x\to 0}{\text{Lt}}\ \dfrac{1}{\cos x}=1.$

But when $m-n<0$, *i.e.* $-$ive, then

$$I=\int_0^{\pi/2}\frac{1}{x^{n-m}}\left(\frac{x}{\sin x}\right)^n dx.$$

Here 0 is a point of infinite discontinuity.

Choose $\phi(x)=\frac{1}{x^{n-m}}$ and $\int_0^{\pi/2}\frac{dx}{(x-0)^{n-m}}$ is convergent if and only if $n-m<1$ or $n<(1+m)$.

$$\underset{x\to 0}{\text{Lt}}\frac{f(x)}{\phi(x)}=\underset{x\to 0}{\text{Lt}}\left(\frac{x}{\sin x}\right)^n=1.$$

Hence both the integrals converge or diverge together, but since one of them converges when $n<(m+1)$, therefore the other also converges when $n<(m+1)$.

§ 9. μ-test for convergence of $\int_a^b$ f(x) dx.

Let us suppose that $x=a$ is a point of infinite discontinuity and let $f(x)$ be bounded and integrable in the arbitrary interval $(a+\epsilon, b)$ where $0<\epsilon<b-a$. Then if there is +ive number $\mu<1$ and $\underset{x\to a+0}{Lt}(x-a)^\mu f(x)$ exists and is finite, then $\int_a^b f(x)\,dx$ is absolutely convergent.

If $\mu\geqslant 1$ and $\underset{x\to a+0}{Lt}(x-a)^\mu f(x)$ is finite, then $\int_a^b f(x)\,dx$ diverges. Also if $\underset{x\to a+0}{Lt}(x-a)^\mu f(x)$ is $\pm\infty$, then also $\int_a^b f(x)\,dx$ diverges.

In case $x=b$ is a point of infinite discontinuity, then we should evaluate $\underset{x\to b-0}{Lt}(b-x)^\mu f(x)$ and other conditions remaining the same.

Ex. 1. *Test for convergence the integral*

$$\int_0^1\frac{dx}{x^{1/2}(1-x)^{1/3}}.\qquad\text{(Sagar 64, 62 ; Agra 56)}$$

Evidently $x=0$ and $x=1$ are both points of infinite discontinuity

$$\therefore \quad I=\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}=\int_0^a \frac{dx}{x^{1/2}(1-x)^{1/3}}+\int_a^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$$

where $0<a<1$. ...(1)

Now $\int_0^a \frac{dx}{x^{1/2}(1-x)^{1/3}}$. Here $x=0$ is a point of infinite discontinuity.

$\underset{x\to 0+0}{\text{Lt}} x^{1/2} \frac{dx}{x^{1/2}(1-x)^{1/3}}=1$, *i.e.* finite and $\mu=\frac{1}{2}$ *i.e.* <1. Hence by μ-test the integral is convergent.

For $\int_a^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$, $x=1$ is a point of infinite discontinuity.

$$\underset{x\to 1-0}{\text{Lt}} (1-x)^{1/3}\cdot\frac{1}{x^{1/2}(1-x)^{1/3}}=\underset{x\to 1-0}{\text{Lt}} \frac{1}{x^{1/2}}=\underset{h\to 0}{\text{Lt}} \frac{1}{(1-h)^{1/2}}=1$$

But $\mu=\frac{1}{3}$, *i.e.* <1 and hence this integral is also convergent.

Hence the given integral being the sum of two convergent integrals is also convergent.

Similarly we can show that the following integrals are also convergent :

$$\int_0^1 \frac{dx}{\sqrt{[x(1+x)]}}, \int_0^1 \frac{dx}{\sqrt{x}.(1+x)}.$$

But $\int_0^1 \frac{dx}{x(1+x)}$ will be divergent because in this case $\mu=1$.

Ex. 2. *Test the convergence of* $\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\,dx$.

(Rajputana 63)

Here again both at $x=0$ and $x=2$, there is an infinite discontinuity

$$\therefore \quad I=\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\,dx=\int_0^a \frac{\log x}{\sqrt{(2-x)}}\,dx+\int_a^2 \frac{\log n}{\sqrt{(2-x)}}\,dx,$$

$0<a<2$. ..(1)

Convergence at x=0. $f(x)=\dfrac{\log x}{\sqrt{(2-x)}}$.

$$\underset{x\to 0}{\text{Lt}}\ x^{\mu}\frac{\log x}{\sqrt{(2-x)}}=0 \text{ if } \mu>0.$$

Hence if we choose μ between 0 and 1, the integral $\int_0^a f(x)\,dx$ is convergent by μ-test.

Convergence at x=2. $\int_a^2 \dfrac{\log x}{\sqrt{(2-x)}}$, $x=2$, is a point of infinite discontinuity.

$$\underset{x\to 2-0}{\text{Lt}}(2-x)^{1/2}\frac{\log x}{\sqrt{(2-x)}}=\underset{h\to 0}{\text{Lt}}\log(2-h)=\log 2.$$

But $\mu=\frac{1}{2}$, i e. <1 and hence $\int_a^2 \dfrac{\log x}{\sqrt{(2-x)}}\,dx$ is convergent.

Hence the given integral being the sum of two convergent integrals is also convergent

Ex. 3. *Test the convergence of* $\int_0^{\pi/2} \dfrac{2\cos x}{x^n}\,dx$.

(Agra 51, 58)

When **x is $-$ive, i.e. <0,** then the above integral is proper and hence convergent.

Again when n is >0, i.e. $+$ive, then $x=0$ is a point of infinite discontinuity.

$$f(x)=\frac{\cos x}{x^n} \text{ and } \underset{x\to 0}{\text{Lt}}\ x^{\mu}.f(x)=\underset{x\to 0}{\text{Lt}}\ x^{\mu}.\frac{\cos x}{x^n}$$

$$=\underset{x\to 0}{\text{Lt}}\ x^{\mu-n}\cos x=1 \text{ if } \mu=n.$$

But by μ-test the integral is convergent if $0<\mu<1$ and divergent when $\mu\geqslant 1$. Hence the given integral is convergent if $0<n<1$ and divergent when $n\geqslant 1$.

Ex. 4. $\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\, dx$ *converges when* $0 < n < 1$,

$$\frac{\sin x}{x^{1+n}} = \frac{1}{x^n} \cdot \frac{\sin x}{x}.$$

When n is $-$ive, i.e. < 0, then above integral is proper

$$\because \quad \underset{x\to 0}{\text{Lt}} \frac{\sin x}{x} = 1.$$

If n is $+$ive, *i.e.* > 0, then $x=0$ is a point of infinite discontinuity.

$$f(x) = \frac{1}{x^n} \cdot \frac{\sin x}{x} \text{ choose } x^{\mu} = x^n.$$

$$\underset{x\to 0}{\text{Lt}}\, x^n \cdot \frac{1}{x^n} \left(\frac{\sin x}{x}\right) = 1, \text{ i.e. finite}$$

Hence the given integral is convergent when $0 < n < 1$ and divergent when $n \geqslant 1$ by μ-test.

Ex. 5. *Test the convergence of* $\int_0^{\pi/4} \frac{dx}{\sqrt{(\tan x)}}$.

(Agra 52)

$x=0$ is a point of infinite discontinuity.

$$f(x) = \frac{1}{\sqrt{(\tan x)}} = \sqrt{\left(\frac{\cos x}{\sin x}\right)} \text{ and choose } x^{\mu} = x^{1/2}.$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}}\, x^{\mu} f(x) = \underset{x\to 0}{\text{Lt}}\, x^{1/2} \cdot \sqrt{\left(\frac{\cos x}{\sin x}\right)}$$

$$= \underset{x\to 0}{\text{Lt}} \sqrt{\left(\frac{x}{\sin x}\right)}\, \sqrt{(\cos x)} = 1$$

Since the limit is finite and $\mu = \frac{1}{2}$, *i.e.* < 1, hence the given integral by μ-test is convergent.

Ex. 6. *Test the convergence of* $\int_0^1 x^{n-1} e^{-x}\, dx$.

(Jiwaji 66 ; Gauhati Hons. 63)

In case $n-1 \geqslant 0$, *i.e.,* $n \geqslant 1$, *then the given integral is* proper and hence convergent. But if $n-1 < 0$, *i.e.* $n < 1$,

then it is improper and $x=0$ is a point of infinite discontinuity.

$$f(x)=x^{\alpha-1}e^{-x}\text{ ; choose } x^{\mu}=x^{1-\alpha}$$

$$\underset{x\to 0}{\text{Lt}}\,(x^{\alpha-1}e^{-x}).x^{1-\alpha}=1,\ \textit{i.e}, \text{ finite.}$$

Hence by μ-test the given integral is convergent if μ, *i.e.* $1-\alpha<1$ and divergent if μ *i.e.* $1-\alpha\geqslant 1$ or $\alpha>0$, convergent and $\alpha\leqslant 0$, divergent.

But $\alpha<1$; $\therefore$ $0<\alpha<1$, convergent and $\alpha\leqslant 0$, divergent.

Ex. 7. Convergence of Gamma function.

$\Gamma n=\int_0^{\infty} x^{n-1}e^{-x}\,dx$ *is convergent only when* $n>0$.

(Karnatak 62, 64 ; Agra 60 ; Rajputana 51, 62, 63 ; Pb. 53, 56 ; Vikram 63)

1st Case. Let $n\geqslant 1$, *i.e.* $(n-1)$ is +ive, then

$$\int_0^{\infty} x^{n-1}e^{-x}\,dx=\int_0^{a} x^{n-1}e^{-x}\,dx+\int_a^{\infty} x^{n-1}e^{-x}\,dx.$$

The first integral is proper and hence convergent and we therefore consider the convergence of

$$\int_a^{\infty} x^{n-1}e^{-x}\,dx.\quad \text{Now } f(x)=x^{n-1}e^{-x}.$$

$$\underset{x\to\infty}{\text{Lt}}\, x^{\mu} f(x)=\underset{x\to\infty}{\text{Lt}}\,\frac{x^{\mu+n-1}}{e^{x}}$$

$$=\underset{x\to\infty}{\text{Lt}}\,\frac{x^{\mu+n-1}}{1+x+\frac{x^2}{2\,!}+\dots}=0$$

for all values of μ and n.

Choosing $\mu>1$, we can say by § 9 P. 319 that $\int_a^{\infty} x^{n-1}e^{-x}\,dx$ is convergent.

Hence $\int_0^\infty x^{n-1}e^{-x}\,dx$ is also convergent, being the sum of two convergent integrals.

2nd Case. $0 < n < 1$, *i e*, $n-1$ is $-$ive.

$$\int_0^\infty x^{n-1}e^{-x}\,dx=\int_0^a x^{n-1}e^{-x}\,dx+\int_a^\infty x^{n-1}e^{-x}\,dx.$$

The second integral is convergent for all values of n as *shown in the 1st case.*

But $\int_0^a x^{n-1}e^{-x}\,dx$, $0 < n < 1$, *has a point of infinite* discontinuity at $x=0$ as $n-1$ is $-$ive.

$$f(x)=x^{n-1}e^{-x}.$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}}\; x^{\mu} f(x)=\underset{x\to 0}{\text{Lt}}\; \frac{x^{\mu+n-1}}{e^{x}}=1,$$

if $\mu+n-1=0$ or $\mu=1-n$, but n lies between 0 and 1, and hence μ is less than 1. Therefore by § 9 P. 319, the integral is convergent.

$$\therefore \quad \int_0^\infty x^{n-1}e^{-x}\,dx,\ 0 < n < 1, \text{ is convergent.}$$

3rd Case. $n \leqslant 0$.

$$\int_0^\infty x^{n-1}e^{-x}\,dx=\int_0^a x^{n-1}e^{-x}\,dx+\int_a^\infty x^{n-1}e^{-x}\,dx.$$

The second integral is convergent for all values of n as shown in case I.

But $\int_0^a x^{n-1}e^{-x}$ $n \leqslant 0$, has a point of infinite discontinuity at $x=0$.

$$f(x)=x^{n-1}e^{-x}.$$

$$\therefore \quad \underset{x\to 0}{\text{Lt}}\; x^{\mu} f(x)=\underset{x\to 0}{\text{Lt}}\; \frac{x^{\mu+n-1}}{e^{x}}=1,$$

if $\mu+n-1=0$ or $\mu=1-n$, but n is $-$ive.

$\therefore\ \mu > 1$, and hence the integral is divergent by μ-test of P. 319 § 9.

Hence $\int_0^\infty x^{n-1} e^{-x}\, dx,\ n \leqslant 0$, is divergent.

From above we can say that the given integral is convergent when $n > 0$

Ex. 8. *Examine the convergence of* $\int_0^1 x^{n-1} \log x\, dx$

. (Vikram 64)

1st Case. $0 > n > 1$.

We know that $\underset{x\to 0}{\text{Lt}}\ x^r \log x = 0$, when $r > 0$.

Therefore the given integral is proper and hence convergent, when $n-1 > 0$, *i.e.* $n > 1$.

2nd Case. $n = 1$.

$I = \int_0^1 \log x\, dx$, $x=0$, is a point of infinite discontinuity.

$$\therefore\ \int_0^1 \log x\, dx = \underset{\epsilon\to 0}{\text{Lt}} \int_\epsilon^1 \log x\, dx = \underset{\epsilon\to 0}{\text{Lt}} \left(x \log x - x\right)_\epsilon^1$$

$$= \underset{\epsilon\to 0}{\text{Lt}}\ [(0-1)-(\epsilon \log \epsilon - \epsilon)] = -1.$$

Hence the integral is convergent.

3rd Case. $0 < n < 1$, *i.e.* $n-1$ is $-$ive.

$f(x) = x^{n-1} \log x$; $\therefore\ \underset{x\to 0}{\text{Lt}}\ x^\mu f(x) = \underset{x\to 0}{\text{Lt}}\ x^{\mu+n-1} \log x$.

Above limit is zero, if $\mu+n-1 > 0$ or $\mu > 1-n$. But because $0 < n < 1$, we can always choose μ to be between 0 and 1 such that $\mu > 1-n$ and hence by μ-test the integral is convergent.

4th Case. $n \leqslant 0$.

Here $\underset{x\to 0}{\text{Lt}}\ x^\mu f(x) = \underset{x\to 0}{\text{Lt}}\ x^{\mu+n-1} \log x = -\infty$

if $\mu+n-1 \leqslant 0$ or $\mu \leqslant 1-n$.

When $n \leqslant 0$, μ can always be found such that $\mu \geqslant 1$ and hence the integral is divergent in this case.

Hence we conclude that the integral is convergent only when $n > 0$ and divergent when $n \leqslant 0$.

Ex. 9. *Discuss the convergence and divergence of the integral* $\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx.$

(Rajputana 58 ; Karnatak 64 ; Gauhati Hons. 63)

1st Case. $\alpha \geqslant 1$.

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \int_0^a \frac{x^{\alpha-1}}{1+x}\,dx + \int_a^\infty \frac{x^{\alpha-1}}{1+x}\,dx$$

The first integral is clearly proper when $\alpha \geqslant 1$ and hence convergent and $\int_a^\infty \frac{x^{\alpha-1}}{1+x}\,dx.$

Choose $\mu = 1-(\alpha-1) = 2-\alpha$.

$$\underset{x\to\infty}{\text{Lt}}\; x^\mu f(x) = \underset{x\to\infty}{\text{Lt}}\; x^{2-\alpha}\cdot\frac{x^{\alpha-1}}{1+x} = \text{Lt}\;\frac{x}{1+x} = 1.$$

But $\mu = 2-\alpha$, where $\alpha \geqslant 1$; $\therefore$ $\mu \leqslant 1$ and hence the integral is divergent by μ-test of § 9 P. 319.

2nd Case. $\alpha < 1$.

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \int_0^a \frac{x^{\alpha-1}}{1+x}\,dx + \int_a^\infty \frac{x^{\alpha-1}}{1+x}\,dx.$$

$\int_0^a \frac{x^{\alpha-1}}{1+x}$ when $\alpha < 1$ has a point of infinite discontinuity at $x=0$

$$\therefore \underset{x\to 0}{\text{Lt}}\; x^\mu f(x) = \frac{x^{\mu+\alpha-1}}{1+x} = 1,$$

if $\mu+\alpha-1=0$ or $\mu = 1-\alpha$. If we choose $0 < \alpha < 1$, then

μ lies between 0 and 1 and hence the integral is convergent by μ-test of § 9 P. 319.

Also $\int_a^\infty \frac{x^{\alpha-1}}{1+x}\,dx$; choose $\mu = 1-(\alpha-1) = 2-\alpha$

$$\underset{x\to\infty}{\text{Lt}}\ x^\mu f(x) = \underset{x\to\infty}{\text{Lt}}\ x^{2-\alpha}\cdot\frac{x^{\alpha-1}}{1+x} = \text{Lt}\ \frac{x}{1+x} = 1.$$

But $\mu = 2-\alpha$, where $\alpha < 1$; $\therefore\ \mu > 1$ and hence the integral is convergent. Therefore the given integral is convergent if $0 < \alpha < 1$.

GAUHATI UNIVERSITY (Hons.) PAPERS

1965

1. (*a*) How do you define "infinite integrals" ? Deduce from your definition that

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}, \; a > 0$$

(*b*) Discuss the covergence of the following integrals :—

(*i*) $\int_0^\infty e^{-x} x^{n-1} \, dx.$

(*ii*) $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx.$

2. Evaluate any three of the following :—

(*i*) $\int_0^\infty e^{-x^2} \, dx$;

(*ii*) $\int_0^\infty \frac{\sin x}{x} \, dx$.

(*iii*) $\int_0^\infty \frac{\sin ax \sin bx}{x} \, dx = \frac{1}{2} \log \frac{a+b}{a-b} \; (a > b > 0)$;

(*iv*) $\int_0^{\pi/2} \log \sin x \, dx$;

(*v*) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$.

3. (*a*) Denoting the integral $\int_0^\infty e^{-x} x^{n-1} \, dx$ by $\Gamma(n)$ whenever it converges, show that

$$\Gamma(n) = (n-1) \, \Gamma(n-1)$$

and deduce the value of $\Gamma(n)$ when n is a positive integer.

(*a*) Prove that $\int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta$

$$= \Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right) \Big/ \left\{2\Gamma\left(\frac{m+n}{2}+1\right)\right\}$$

4. (*a*) Explain and illustrate the process of differentiation under the sign of integration

(*b*) Prove that

$$\int_0^{\pi/2} \log (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)\, d\theta = \pi \log \frac{\alpha+\beta}{2}$$

5. (*a*) Evaluate $\iint_R y\, dx\, dy$, where R is the first quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(*b*) Find by triple integration the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes.

6. (a) Assuming that a function $f(x)$ defined in the interval $(-\pi, \pi)$ can be represented as a trigonometrical series, $a_0+(a_1 \cos x+b_1 \sin x)+(a_2 \cos 2x+b_2 \sin 2x)+\ldots$ where a_0, a_1, b_1, etc., are constants and further that we can integrate the series term by term, prove that

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

and

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

(*b*) Show that the Fourier series corresponding to x in in the interval $(-\pi, \pi)$ is

$$2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots\right)$$

1967

1. (a) Show that if $a > -1$, $b > 0$, $bc > a+1$,

$$\int_0^{\infty} \frac{x^a\, dx}{(1+x^b)^c}=\frac{1}{b} B\left(\frac{a+1}{b},\ c-\frac{a+1}{b}\right)$$

$$=\frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c-\frac{a+1}{b}\right)}{b\,\Gamma(c)}.$$

(b) If $0 < p < 1$, show that

$$\int^{\pi/2} \tan^p x \, dx = \int_0^{\pi/2} \cot^p x \, dx = \frac{1}{2} B\left(\frac{1+p}{2}, \frac{1-p}{2}\right).$$

2. (a) Prove the following theorem :

If $\phi(x)$ is a positive monotonic decreasing function, continuous in the interval $c \leqslant x < \infty$, and if $\lim\limits_{x \to \infty} \phi(x) = 0$, then the integrals

$$\int_c^\infty \phi(x) \sin x \, dx, \quad \int_c^\infty \phi(x) \cos(ax+b) \, dx, \; a \neq 0$$

converge.

(b) Test for convergence the following integrals

(i) $\displaystyle\int_0^\infty \frac{x \cos x}{a^2 + x^2} \, dx,$

(ii) $\displaystyle\int_0^\infty \cos x^2 \, dx.$

3. (a) If $\phi(x)$ and $\phi'(x)$ are continuous and integrable over an arbitrarily large interval (λ, μ), $\lambda > 0$, and that the integrals of $\phi(x)$ and $\phi'(x)$ converge or *oscillate finitely* and if $a > 0$, $b > 0$, show that

$$\int_0^\infty \left\{ \frac{\phi(ax) - \phi(bx)}{x^2} - (a-b) \frac{\phi'(x)}{x} \right\} dx$$
$$= \phi(0) \{ b \log b - a \log a + a - b \}.$$

(b) Prove that

$$\int_0^\infty \frac{\sin ax \sin bx}{x^2} \, dx = \frac{\pi}{2} b, \; a > b > 0$$

4. (a) Prove that a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ defines a *continuous function for all values of x in any closed interval (a, b) that is interior to the interval of convergence of the series.*

(b) Differentiate term by term the power series in x for $\sin x$ and thus obtain the power series in x for $\cos x$. What is the interval of convergence of the resulting series ?

5. Evaluate the following integrals :

(i) $\int_0^\infty \frac{e^{-ax} \sin mx}{x} dx$, (ii) $\int_0^1 \frac{\log x \, dx}{1+x}$,

(iii) $\int_0^\infty \frac{e^{-x}-e^{-nx}}{x} dx$.

6. (a) Show that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ay$ is

$$a^3\left(\frac{2\pi}{3}-\frac{8}{9}\right).$$

(b) Evaluate $\iint \frac{\sqrt{(a^2b^2-b^2x^2-a^2y^2)}}{\sqrt{(a^2b^2+b^2x^2+a^2y^2)}} dx\, dy$, the area of integration being the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

7. (a) State the conditions on $f(x)$ that are sufficient to permit its representations by a Fourier's series.

(b) Show that

$$\log\left(2 \sin \frac{x}{2}\right)=-\sum_{n=1}^{\infty} \frac{\cos nx}{n} \text{ if } 0 < x < \pi.$$

RAJASTHAN M.A. and M. Sc. (Prev.) PAPERS

1963

1. (a) Test the convergence of the following integrals ;

(i) $\int_0^\infty e^{-x} x^{n-1}\, dx$.

(ii) $\int_0^2 \frac{\log x}{\sqrt{(2-x)}} dx$.

(b) Find the values of :

(i) $\int_0^1 \frac{z^a - z^{-a}}{1-z}\, dz.$

(ii) $\int_0^\infty \frac{\log (1+a^2x^2)}{1+b^2x^2}\, dx.$

2. (a) Find the value of

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right). \quad \Gamma\left(\frac{n-1}{n}\right),$$

n being an integer.

(b) Evaluate

$$\int_0^1 \frac{x^2\, dx}{\sqrt{(1-x^4)}} \times \int_0^1 \frac{dx}{\sqrt{(1+x^4)}}.$$

(c) Evaluate the integral

$$\iiint \frac{dx\, dy\, dz}{\sqrt{(a^2-x^2-y^2-z^2)}},$$

the integrals being extended to all positive values of the variables for which the expression is real.

3. (a) Transform the integral

$$\int_0^{\pi/2}\int_0^{\pi/2} \sqrt{\left(\frac{\sin \phi}{\sin \theta}\right)}\, d\phi\, d\theta$$

by the substitution

$$x = \sin \phi \cos \theta,$$
$$y = \sin \phi \sin \theta.$$

and show that its value is π.

(b) Prove that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ is $\frac{2}{9}(3\pi-4)\, a^3$.

4 Define a surface integral and prove that

$$\iiint_W \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) dx\, dy\, dz = -\iint_S (lu+mv+nw)\, dS,$$

where u, v, w are single-valued functions of x, y, z, their derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$ are continuous throughout, the volume

W bounded by a surface S and l, m, n are the direction-cosines of the normal at a point $P(x, y, z)$ on S and dS is the element of the surface S at (x, y, z).

If u, v, w satisfy the conditions of Green's theorem, if K is a surface inside W bounded by a closed curve C, under what conditions will the surface integral

$$\iint_K (lu+mv+nw)\, dS$$

depend solely on the curve C and not on the particular surface K on which C lies ?

1965

1. (a) A bounded function $f(x)$ is integrable over $[a, b]$ and M, m are bounds of $f(x)$, show that

$$m(b-a) \leqslant \int_a^b f(x)\, dx \leqslant M(b-a),$$

(b) Test the convergence of the following integrals .—

(i) $\int_{-\infty}^{\infty} e^{-x^2}\, dx$. (ii) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x}\, dx$.

2. (a) Prove any two of the following :—

(i) $\int_0^a \frac{dx}{(a^6-x^6)^{1/6}}=\frac{\pi}{3}$. (ii) $\int_{-\infty}^{\infty} \cos\frac{\pi x^2}{2}\, dx=1$.

(iii) $\int_0^{\pi/2} \log(1+\cos\theta\cos x)\frac{dx}{\cos x}=\frac{\pi^2-4\theta^2}{8}$.

(b) Prove that

$$\iiint \sqrt{\left(\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}\, dx\, dy\, dz=\frac{\pi}{8}\left\{B\left(\frac{3}{4}, \frac{1}{2}\right)-B\left(\frac{5}{4}, \frac{1}{2}\right)\right\},$$

the integral being taken over all positive values of x, y, z, such that $x^2+y^2+z^2 \leqslant 1$.

3. (a) Prove that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ is

$$\tfrac{2}{9}a^3(3\pi-4).$$

(b) Define line integral and show that the value of

$$\int_C (x^2+y^2)\,dx \text{ and } \int_C (x^2+y^2)\,dy$$

where C is the arc of the parabola $y^2=4ax$ between $(0, 0)$, $(a, 2a)$ is $\frac{7}{3}a^3$ and $\frac{46}{15}a^3$ respectively.

4. (a) *Give the statement of the two dimensional form* of Green's Theorem.

(b) Transform the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\sqrt{\left(\frac{\sin\phi}{\sin\theta}\right)}\,d\phi\,d\theta,$$

by the substitution $x=\sin\phi\cos\theta$, $y=\sin\phi\sin\theta$ and find its value.

SAGAR UNIVERSITY PAPERS

1963

1. (a) Define Gamma Function, and prove that

$$\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

(b) *Show that* $\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-b^2/a^2}$

2. (a) Change the order of integration in the double integral $\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V\,dx\,dy$, V being a function of x and y.

(b) Test the convergence of the following integrals :

(i) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$,

(ii) $\int_0^\infty \sin x^2\,dx$.

3. (a) Define a Fourier series.

Find the series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi < x < \pi$, when

$$f(x)=0 \quad -\pi < x \leq 0,$$
$$f(x)=\tfrac{1}{4}\pi x, \quad 0 < x < \pi.$$

Hence show that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

(b) Evaluate the integral $\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$, where x, y, z are always positive but limited by the condition

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r < 1.$$

1964

1. (a) Evaluate $\int_0^\infty e^{-x^2}\,dx$ and deduce that

$$\int_0^\infty e^{-x^2}x^{2n}\,dx=\frac{1.3.5\ldots(2n-1)}{2^{n+1}}\sqrt{\pi}.$$

(b) Prove that

$$\int_0^a \frac{x^{l-1}(a-x)^{m-1}\,dx}{(a+bx)^{l+m}}=\frac{B(l, m)}{a(1+b)^l}.$$

2. (a) By changing the order of integration evaluate the integral $\quad \int_0^\infty \int_x^\infty (ye^y)^{-1}\,dx\,dy.$

(b) Show that

$$\iiint \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}=\frac{\pi^2}{8}.$$

the integral being extended to all positive values of the variables for which the expression $\sqrt{(1-x^2-y^2-z^2)}$ is real.

3. (a) Test the convergence of one of the following integrals :

(i) $\int_0^1 \frac{dx}{x^2(1+x)^2}$, (ii) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$.

(b) State and prove Green's Theorem.

4. (a) The space enclosed by the surface

$$\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=1$$

is full of matter whose density at any point is given by $\rho=(xyz)^3$. Find the whole mass.

(b) Expand in a series of sines and cosines of multiples of x, the funciion

$$f(x)=x-\pi \quad \text{when} \quad -\pi < x < 0,$$
$$f(x)=\pi-x \quad \text{when} \quad 0 < x < \pi.$$

What is the sum of the series for $x=\pm\pi$ and $x=0$? Hence show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}.$$

1966

1. Test for convergence the integrals :

(i) $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$, (ii) $\int_0^{\pi/2} \sin^{m-1} x \cos^{n-1} x\,dx$.

2. (a) Define Beta and Gamma functions and establish the relation between them.

(b) Evaluate $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx$.

3. Prove that

(i) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi}{8}\log 2$,

(ii) $\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^4)}}\times\int_0^1 \frac{dx}{\sqrt{(1+x^4)}}=\frac{\pi}{4\sqrt{2}}$,

(iii) $\int_0^\infty e^{-a^2x^2}\cos 2bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-b^2/a^2}$.

4. (a) Change the order of integration in

$$I=\int_0^a\int_0^a \frac{\cos y\,dx\,dy}{\sqrt{\{(a-x)(a-y)\}}}$$

and hence evaluate it.

(b) Find the centre of gravity of the mass of the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose density is constant.

5. (a) Obtain the Fourier series of the function

$$f(x)=x-\pi, \text{ when } -\pi < x < 0$$

and

$$f(x)=\pi-x, \text{ when } 0 < x < \pi.$$

Deduce that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

(b) If $0 \leqslant a_n < 1$, prove that the product $\Pi(1-a_n)$ and the series Σa_n converge or diverge together.

JIWAJI UNIVERSITY PAPER

1966

1. (a) Prove that

$$\int_0^\infty x^{a-1}e^{-x}\,dx \text{ converges if } a > 0.$$

(b) With usual notation prove that

$$B(m, n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

6. Evaluate the following integrals

(i) $\int_0^\infty e^{-x^2}\,dx.$ (ii) $\int_0^\infty \frac{x^{n-1}}{1+x}\,dx \quad (0 < n < 1)$

(*iii*) $\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$, such that $x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$ and $x+y+z \leqslant 1$.

2. (a) Find the mass of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the volume density being given by $\rho=\mu xyz$.

(*b*) Change the order of integration in

$$\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} V \, dx \, dy,$$

where V is a function of x and y

3 (a) Show that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ is

$$a^3\left(\frac{2\pi}{3}-\frac{8}{9}\right).$$

(b) Find a series of sines and cosines of multiples of x which will represent $x+x^2$ in $-\pi < x < \pi$ and deduce that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

VIKRAM M.A. and M.Sc. (Prev.) UNIVERSITY PAPERS 1964

1. (a) Defining an 'improper integral', show by means of an example that even when improper integral does not exist, its 'principal value' may exist.

(b) Discuss the convergence and divergence of the integral

$$\int_0^1 x^{n-1} \log x \, dx,$$

(c) Evaluate $\displaystyle\int_0^\infty e^{-x^2} \, dx.$

2. (a) Define Beta and Gamma functions and establish a relation between the two.

(b) Evaluate (i) $\displaystyle\int_0^{\pi/2} \sin^{p-1} z \, dz$;

(ii) $\int_0^\infty \cos (bz^{1/n})\, dz.$

3. Express the integral $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} V_{(x, y)}\, dx\, dy$ by changing its order of integration

(b) State and prove Dirichlet's theorem on the evaluation of multiple integrals.

4. (a) Find the volume bounded by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

(b) Obtain the Fourier series which will represent $x+x^2$ in the interval $-\pi < x < \pi$.

1965

1. (a) What are proper and improper integrals ? State the necessary and sufficient conditions for the convergence of improper integrals.

(*b*) Test for convergence the integral

$$\int_0^{\pi/2} \sin^{m-1}x \cos^{n-1}x\, dx.$$

2. (a) Define the Beta function $B(m, n)$ and show

$$B(m, n)=B(n, m).$$

Also, show that

$$2B(m, n)=\int_0^1 \frac{y^{n-1}+y^{m-1}}{(1+y)^{m+n}}\, dy.$$

(b) Find the value of $\int_0^\infty e^{-x^2}\, dx.$

Hence, show that

$$\int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx=\int_0^\infty \frac{\sin x}{\sqrt{x}}\, dx=\sqrt{\left(\frac{\pi}{2}\right)}.$$

(c) Show that

$$\int_0^\infty \frac{1}{x}\,(e^{-ax}-e^{-bx})\, dx=\log\left(\frac{b}{a}\right)\ ;\ a>0,\ b>0.$$

3 (a) Prove that

$$\int_0^\infty \int_0^\infty \phi(a^2x^2+b^2y^2)\,dx\,dy=\frac{\pi}{4ab}\int_0^\infty \phi(t)\,dt.$$

(b) Discuss Liouville's extension of Dirichlet's theorem on the evaluation of multiple integrals.

4 (a) Define Fourier's coefficients.

Find a series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi < x < \pi$, when

$$f(x)=0, \; -\pi < x \leqslant 0;$$
$$=\tfrac{1}{4}\pi x, \; 0 < x < \pi.$$

Hence, show that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$

(b) Find the surface area of the portion of the cylinder $x^2+z^2=16$ lying inside the cylinder $x^2+y^2=16$

INDORE UNIVERSITY PAPER

1966

1. (a) Give Eulers's definition of Beta function. Discuss the convergence of this Eulerian integral.

(b) Test for convergence the integral $\int_0^\infty \frac{\sin(x^\beta)}{x^\alpha}\,dx.$

2 (a) Prove that

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2})=\frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n), \text{ where } n > 0.$$

(b) Evaluate :—

(i) $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$

(ii) $\int_0^\infty \frac{\sin mx}{x}\,dx$ $(m < 0)$.

3. (a) Change the order of integration in

$$\int_0^{a/2} \int_{x^2/a}^{x-x^2/a} V\,dx\,dy.$$

(b) Evaluate :—

$$\iint \cdots \int \frac{dx_1\,dx_2 \ldots dx_n}{\sqrt{(a^2-x_1^2-x_2^2-\ldots-x_n^2)}},$$

where the variables are positive with the condition

$$x_1^2+x_2^2+\ldots+x_n^2 \not> a^2.$$

4. (a) Find the volume of the region above the xy-plane bounded by the paraboloid

$z=x^2+y^2$ and the cylinder $x^2+y^2=a^2$.

(b) Prove that for $0 \leqslant x \leqslant \pi$

$$x(\pi-x)=\frac{1}{6}\pi^2-\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 6x}{3^2}+\ldots\right).$$

Hence, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}.$$

AGRA UNIVERSITY PAPERS

1965

1. (a) Prove that, if $a>0$, the integral

$$\int_a^{\infty} \frac{dz}{z^n}$$

is convergent when $n>1$ and is divergent when $n \leqslant 1$.

Hence show that $\displaystyle\int_b^{\infty} \frac{x^{3/2}\,dx}{\sqrt{(x^4-a^4)}}$

is divergent, b being greater than a.

(b) Find the Fourier's series for $f(x)$ in the interval

$(-\pi, \pi)$, where

$$f(x)=\pi+x \text{ when } -\pi < x < 0,$$
$$f(x)=\pi-x \text{ when } \quad 0 < x < \pi.$$

2 Evaluate :—

(i) $\int_0^1 \frac{z^a-z^{-a}}{1-z}\,dz$;

(ii) $\int_0^{\pi/2} \log(1+\cos\theta\cos x)\,\frac{dx}{\cos x}$,

(iii) $\int_0^\infty \cos(bz^{1/n})\,dz$.

3 Transform into polar co-ordinates and integrate

$$\iint \frac{\sqrt{(1-x^2-y^2)}}{\sqrt{(1+x^2+y^2)}}\,dx\,dy,$$

the integral being extended to all positive values of x and y subject to $x^2+y^2 \leqslant 1$

(b) Show that $\iiint \frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}=\frac{\pi^2}{8}$,

the integral being extended to all positive values of the variables for which the expression is real

1966

1 Evaluate :

(i) $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$. (ii) $\int_0^\infty e^{-a^2x^2}\,dx$.

(iii) $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$, n being an integer.

2. (a) Find the volume enclosed by

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n+\left(\frac{z}{c}\right)^n=1.$$

(b) Evaluate the integral

$$\iiint \frac{dx\,dy\,dz}{\sqrt{(a^2-x^2-y^2-z^2)}},$$

the integral being extended to all positive values of the variables for which the expression is real

6. (a) Test the convergence of the integral

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\, dx,$$

where m and n are positive integers

(b) Obtain Fourier's Series for the expansion of $f(x)=x \sin x$ in the interval $(-\pi, \pi)$. Hence deduce that

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}\cdots$$

1967

1. (a) Point out the peculiarities of the integrand which make it necessary to consider the equation of convergence of the integral

$$\beta(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

and show that it is convergent when m and n are positive

(b) Integrate :

(i) $\int_0^\infty \cos (bz^{1/n})\, dz$. (ii) $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx$.

2. (a) Find the value of

$$\iint\cdots\int dx_1\, dx_2\, dx_3 \ldots dx_n,$$

extended to all positive values of the variables subject to the condition $x_1^2+x_2^2+x_3^2+\ldots+x_n^2 < R^2$.

(b) Obtain a series of sines and cosines of multiples of x which will represent $f(x)$ in the interval $-\pi < x < \pi$, when

$$f(x)=0,\ -\pi < x < 0,$$
$$=\tfrac{1}{4}\pi x,\ 0 < x < \pi.$$

Hence deduce that $\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}\cdots$

3. (a) Change the order of integration in

$$\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} f(x, y)\, dx\, dy.$$

Hence evaluate

$$\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} \frac{\phi'(y)\,(x^2+y^2)\,x\,dx\,dy}{\sqrt{[4a^2x^2-(x^2+y^2)^2]}}.$$

(b) Find the volume of the sphere $x^2+y^2+z^2=a^2$ included within the cylinder $x^2+y^2=ax$.